THEORIES OF EQUILIBRIUM FIGURES OF A ROTATING HOMOGENEOUS FLUID MASS

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YUSUKE HAGIHARA
University of Tokyo

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Foreword

The opening of the space age had the effect of reviving interest in the field of celestial mechanics, which had been somewhat neglected in the first half of the 20th century. Stimulated by the early artificial earth satellites, the revival began with the simplest problems of spherical astronomy and orbit theory; then it extended into problems of special and finally general perturbations. During the last few years the problem of the figures of the planets and its connection with their rotations has come to the fore, partly because of its relation to the shape of the Earth and partly because of the idea brought forward by D. U. Wise and others that the Moon may really have come out of the Earth.

Just as it was necessary to rediscover and reprint the great treatise of Tisserand from the 1890's for many of the problems of celestial mechanics, so also, I feel, it will be useful to bring out this summary of the problems of the figures of rotating bodies as it stood in the mid 1930's. Hagihara, whom I am proud to count as a friend, has done the scientific world a great service in bringing these scattered works together, summarizing them, and presenting their consequences.

The work of Lyttleton, Chandrasekhar, Roberts, Levinson, and others has recently further advanced the subject. Nevertheless, this comprehensive and thoughtful review, by a powerful mathematician, of the situation as it stood in 1935 will be of great value to the new student of the subject.

John A. O'Keefe
Goddard Space Flight Center
November 1969

Preface

These are the lectures delivered before the staffs of the Smithsonian Astrophysical Observatory. Since the time for the lectures was limited, I could not treat the equilibrium figures of a *heterogeneous* rotating fluid and the difficult problem of the stability of such equilibrium figures was also left out. Poincaré's theory of dynamical tides is included as Appendix A. Both materials are taken from my lecture notes at the University of Tokyo.

On this occasion I would like to express my gratitude to Dr. Fred M. Whipple, director of the Smithsonian Astrophysical Observatory, for his kind invitation and warm hospitality. Appreciation is also due to NASA's Goddard Space Flight Center for accepting this manuscript with its difficult typography.

Y. HAGIHARA
Tokyo

Contents

Introduction

Preliminary theories of the figure of a star as an equilibrium figure of a rotating, homogeneous fluid mass have been treated since Newton by Maupertuis, Maclaurin, Simpson, Clairaut, and Euler. Laplace and Legendre invented a new class of functions and solved Clairaut's problem. Jacobi (1834) discovered an ellipsoidal equilibrium figure with three unequal axes, which was thought to be curious in those days because it rotates, and Liouville, Smith, and Plana proved its existence as an equilibrium figure. Thomson and Tait considered various other figures without proving their existence. It was Tchebycheff (1882) who proposed the problem how the equilibrium figures of a rotating mass of incompressible fluid vary as the rotational speed gradually increases. Liapounov (1884) and Poincaré (1885) solved the problem independently.

When the angular speed is zero, the only stable figure is a sphere. This is Liapounov's theorem. Recent proof of the theorem is based on the rigorous existence theorem of the solution of a corresponding isoperimetric problem in the calculus of variations. As the angular speed increases, a Maclaurin spheroid becomes an equilibrium figure. As it increases further, a Jacobi ellipsoid with three unequal axes appears as a stable equilibrium figure; then, a pear-shaped figure appears as an equilibrium figure. Thus Poincaré, basing his discussion on the expansion in Lamé polynomials, instead of Legendre polynomials as in the case of a sphere, initiated the idea of linear series of such equilibrium figures and the notion of the exchange of stability at the junction of the Maclaurin spheroids and the Jacobi ellipsoids, which he called the *bifurcation point*. As the angular velocity increases from this point, the Jacobi ellipsoid is stable, and the Maclaurin spheroid is no longer stable. A further increase of the angular speed leads to a new bifurcation point where a new linear series for pear-shaped figures appears. But Poincaré concluded that these pear-shaped figures were stable, and Liapounov concluded that they were unstable. Darwin (1902) thought that he had confirmed the Poincaré conclusion by inventing very complicated ellipsoidal harmonics for the discussion, but, since he omitted a term larger in magnitude than the terms he computed, his final conclusion was erroneous.

Liapounov, because he had reached a conclusion opposite to that of Poincaré—the greatest mathematician in those days—reconsidered the problem with great care. Between 1904 and 1914, he published a rigorous proof of the instability in a series of papers in which he confirmed his previous conclusion that a pear-shaped figure is unstable. However, it was necessary to prove a certain inequality, which he did not prove, but thought most probably true.

Jeans (1903) considered a corresponding problem in two dimensions and proved the cylindrical figure corresponding to the pear-shaped to be unstable; he then proceeded to prove the instability of pear-shaped figures by his method of expansion. The convergence of his series was challenged by Baker; however, Baker later found that the series employed by Jeans are simply those of Liapounov in his final method of 1916. Bénèš and Humbert extended Poincaré's work to higher harmonics.

The criterion on which the stability of such equilibrium figures is based is that of Dirichlet. Liapounov and Poincaré based their discussions on different modified forms of this criterion. However, Poincaré's criterion was criticized by Schwarzschild (1896). Liapounov's criterion can answer the question of stability even when Poincaré's cannot, just as in the criterion of stability in particle dynamics (Liapounov, 1949).

Ring-form figures of equilibrium were discussed by Laplace (1859), Maxwell (1859), Mathiessen (1859), Kowalewski (1874), Poincaré (1885), and Lichtenstein (1923). Several detached figures of equilibrium were discussed by Darwin (1906), Lichtenstein (1923, 1933), and his pupils Hölder (1926, 1933) and Garten (1932). Lichtenstein's theory of equilibrium figures is based on a nonlinear integro-differential equation developed by Schmidt and Lichtenstein. The equations on which Lichtenstein's theory is based are those that Liapounov took as his fundamental functional equations.

Liapounov further extended his study to the equilibrium figures of a heterogeneous fluid mass. His manuscript was published after his death by the USSR Academy of Sciences. Now all these papers are published in his collected works.

The question of the figures of the earth and planets is very important in this connection. We must refer to equilibrium figures of a heterogeneous fluid mass such as those recently developed by Dive and Wavre. Moreover, the dynamical figure of the earth is not one of axial symmetry but involves tesseral harmonics, as observations of earth satellites show. Recent developments in the study of the earth's interior reveal a complicated feature with stratification, electric current, and a magnetic field.

The most interesting application of such equilibrium figures is to cosmogony with the supposition that a star might be divided into a system of double stars or have a ring or nebular arms around it by an increase in its angular speed of rotation with constant angular momentum. Laplace, Poincaré, Darwin, and Jeans developed their cosmogonical theories on these assumptions. Recent advances in astrophysics, however, make such theories unsatisfactory unless an essential improvement can be made in the physical aspect of the problem.

Thus the theory of equilibrium figures of a rotating, homogeneous, incompressible fluid mass should be considered a preliminary approach to understanding such natural phenomena.

CHAPTER I

General Properties of Equilibrium Figures

THEOREMS FROM POTENTIAL THEORY

Green's Formula

Let $x(t)$, $y(t)$, $z(t)$ be continuous functions in a closed interval $\langle 0, \pi\rangle$, such that $x(t_1)=x(t_2)$, $y(t_1)=y(t_2)$, and $z(t_1)=z(t_2)$ for $t_1=t_2$. Then the locus of the point with coordinates $x(t)$, $y(t)$, $z(t)$ is continuous. If $x(0)=x(2\pi)$, $y(0)=y(2\pi)$, and $z(0)=z(2\pi)$, then the curve is said to be a *Jordan curve*. Similarly, we define a *Jordan surface* by using two Gaussian parameters μ and ν instead of the parameter t. Any closed surface that can be mapped continuously on the surface of a sphere in a one-to-one correspondence is said to be a Jordan surface. Any closed solid bounded by a set of Jordan surfaces, such that a plane can be drawn at any point on the surface in such a way that the whole solid is on one side of the plane, is called a *convex body*. A *regular* surface is such that the coordinates $x=\varphi(\mu, \nu)$, $y=\psi(\mu, \nu)$, and $z=\chi(\mu, \nu)$, and their first-order derivatives are continuous with respect to the two Gaussian parameters, and that

$$\left[\frac{\partial(\varphi, \psi)}{\partial(\mu, \nu)}\right]^2+\left[\frac{\partial(\varphi, \chi)}{\partial(\mu, \nu)}\right]^2+\left[\frac{\partial(\psi, \chi)}{\partial(\mu, \nu)}\right]^2 \neq 0.$$

Consider a regular region bounded by a regular surface. Let X, Y, Z be continuous both in the solid T and on the bounding surface S and continuously or piecewise differentiable on S. Then we have Green's theorem (otherwise known as Gauss' theorem or the divergence theorem):

$$\int\int\int_T\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)d\tau=\int\int_S(X\ell+Ym+Zn)\,d\sigma$$

or

$$\int\int\int_T \operatorname{div} V d\tau=\int\int_S V_n d\sigma,$$

and Stokes' theorem

$$\int\int_S \left[\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)\ell+\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)m+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)n\right]d\sigma=\int_C (Xdx+Ydy+Zdz).$$

Consider X continuous with its first-order derivatives; then

$$\int\int\int \frac{\partial X}{\partial x}d\tau=\int\int \ell X d\sigma.$$

Put $X=U_1V_1$, $V_1=\partial V/\partial x$; then

$$\int\int\int U_1\frac{\partial^2 V}{\partial x^2}d\tau=\int\int \ell U_1\frac{\partial V}{\partial x}d\sigma-\int\int\int \frac{\partial V}{\partial x}\frac{\partial U_1}{\partial x}d\tau.$$

Adding three such formulas gives Green's formula

$$\int\int\int U_1\Delta V d\tau=\int\int U_1\frac{\partial V}{\partial n}d\sigma-\int\int\int \sum\frac{\partial U_1}{\partial x}\frac{\partial V}{\partial x}d\tau. \tag{1}$$

Interchanging U and V and subtracting these two formulas gives

$$\int\int\int (U\Delta V-V\Delta U)d\tau=\int\int \left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)d\sigma, \tag{2}$$

where ∂n is measured in the direction of the outward normal to the boundary surface. If $\Delta U=0$ and $\Delta V=0$, then

$$\int\int \left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)d\sigma=0. \tag{2a}$$

If we put $U=1$, then

$$\int\int\int \Delta V d\tau=\int\int \frac{\partial V}{\partial n}d\sigma. \tag{3}$$

If $\Delta V=0$, then from equation 3 we get

$$\int\int \frac{\partial V}{\partial n}d\sigma=0.$$

Hence, the integral of the normal derivative of a function, which is harmonic and continuously differentiable, integrated over the boundary of a regular region is zero. Conversely, if the integral of the normal derivative of a function integrated over S is zero, then the function is harmonic in T.

Setting $U=V=U_1$ in equation 1 gives

$$\int\int\int U\Delta U d\tau=\int\int U\frac{\partial U}{\partial n}d\sigma-\int\int\int \sum\left(\frac{\partial U}{\partial x}\right)^2 d\tau. \tag{4}$$

Further, if $\Delta U=0$, then

$$\int\int U\frac{\partial U}{\partial n}\,d\sigma=\int\int\int \sum\left(\frac{\partial U}{\partial x}\right)^2 d\tau>0. \tag{5}$$

The right-hand integral is called the *Dirichlet integral.*

Harmonic Functions

A function U is called *harmonic* when it satisfies the following conditions:

(1) It is continuous together with its first-order derivatives.

(2) Its second-order derivatives exist and are, in general, continuous (if they are discontinuous, the discontinuities are on an algebraic surface—*piecewise continuous*).

(3) $\Delta U=0$.

In Gauss' mean: let U be harmonic and M_0 be a point in a domain and construct a sphere Σ within that domain with M_0 as its center. Let U_0 be the value of U at M_0. Then the mean value of U over the spherical surface with radius r—$\bar{U}=\int U d\sigma/4\pi r^2$—is equal to U_0, whatever the value of r.

In Gauss' theorem, a harmonic function can be neither a minimum nor a maximum inside a domain T except on the bounding surface S.

As a corollary, let g and h be the maximum and minimum respectively of a function U within T; then $h<U<g$ within T, and $h\leqslant U\leqslant g$ on S. If U is zero at infinity, and T extends to infinity, then U is everywhere zero, because both g and h are zero. Conversely, if U is continuous within a closed domain T and the value of U at a point inside T is equal to the mean value of U on the spherical surface with the point as center, then U is harmonic (Koebe, 1906).

From Gauss' theorem expressed by equation 4 we have the following theorems:

(1) If U is harmonic and continuously differentiable inside a regular domain T and vanishes at all points of S, then U vanishes at all points of T.

(2) Any function that is harmonic and continuously differentiable inside a regular domain T is uniquely defined by its value on S.

(3) If the normal derivative on S of a function U, which is harmonic and continuously differentiable inside a regular domain T, is zero at each point on S, then U is constant in T. Such a function can be uniquely defined by the normal derivatives on S apart from an additive constant.

(4) A function U defined by the relation

$$\frac{\partial U}{\partial n}+hU=g,$$

where h, g are continuous on S and $h\geqq 0$, is uniquely determined.

A Newtonian potential is a function that meets the following conditions:

(1) It is continuous at all points in space.

(2) Its first derivatives exist and are continuous both inside and outside of S, but are discontinuous in passing across S; the tangential derivative is continuous while the normal derivative is discontinuous.

(3) $\Delta U=0$ outside S.

(4) ΔU is arbitrary inside S.

(5) $U=0$ at infinity.

The foregoing leads to the theorem that a function which is harmonic in a closed regular domain $T+S$ and is continuously differentiable is a Newtonian potential.

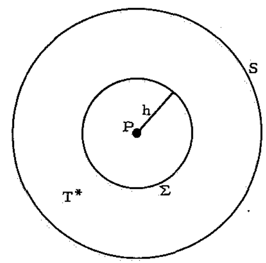
Simple and Double Layers

Set $V=1/r$ in equation 2, where $r=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$, and $P(x, y, z)$ is a point outside T. This function V, as a function of integration-variables (ξ, η, ζ), satisfies the Laplace equation $\Delta V=0$. From equation 2 we obtain

$$\iiint \frac{\Delta U}{r}\, d\tau = -\iint \left\{ U \frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n} \right\} d\sigma.$$

If $P(x, y, z)$ is inside T, this formula fails because P is a singularity of V. Draw a small sphere Σ around P, with radius h as shown at the right. Apply equation 2 to S and Σ and to the space T^* between S and Σ; then

$$\int_T \frac{\Delta U}{r}\, d\tau = -\int_S \left[U \frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n} \right] d\sigma - \int_\Sigma \left[U \frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n} \right] d\sigma.$$



On Σ we have $\partial/\partial n = -\partial/\partial r$ and $d\sigma = h^2 \sin\theta d\theta d\varphi$. Thus

$$\int\int_\Sigma \frac{1}{r}\frac{\partial U}{\partial n}\, d\sigma = h \int\int \frac{\partial U}{\partial n} \sin\theta d\theta d\varphi.$$

Hence,

$$\lim_{h\to 0} \int\int_\Sigma \frac{1}{r}\frac{\partial U}{\partial n} = 0.$$

We have

$$\int\int_\Sigma U \frac{\partial\left(\frac{1}{r}\right)}{\partial n}\, d\sigma = \int\int_\Sigma \frac{U}{r^2}\, d\sigma = \int\int_\Sigma U(h, \theta, \varphi) \sin\theta d\theta d\varphi.$$

Hence

$$\lim_{h\to 0} \int\int_\Sigma U \frac{\partial\left(\frac{1}{r}\right)}{\partial n} = 4\pi U_P.$$

Therefore

$$4\pi U_P = -\int\int_S \left[U \frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n} \right] d\sigma - \int\int\int_T \frac{\Delta U}{r}\, d\tau. \tag{6}$$

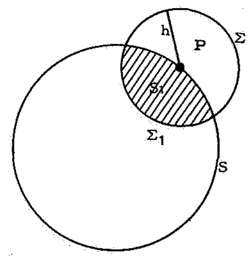
In particular, if $\Delta U=0$, then

$$U_P = -\frac{1}{4\pi}\int\int_S \left[U \frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n} \right] d\sigma. \tag{6a}$$

Next suppose that point P is on S. Draw a sphere Σ with P as its center and with radius h. Let the part of Σ inside S be Σ_1, and the region between S and Σ_1 be S_1. We apply equation 2 to S_1, and make $h \to 0$. As

$$\int\int_{S_1} U d\sigma$$

is integrated over a hemisphere, we have

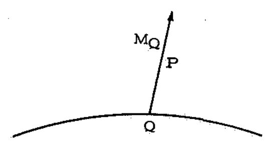


$$U_P = -\frac{1}{2\pi}\int\int_S \left[U\frac{\partial\left(\frac{1}{r}\right)}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n}\right] d\sigma.$$

We say that $\int\int \mu/r \cdot d\sigma$ is the potential of a *simple layer*, and

$$-\int\int \nu\frac{\partial\left(\frac{1}{r}\right)}{\partial n} d\sigma = -\int\int \frac{\nu \cos (r, n)}{r^2} d\sigma$$

is the potential of a *double layer.* As we have seen, any potential consists of the potential of a simple layer of density $(1/4\pi)(\partial U/\partial n)$ and the potential of a double layer of moment $U/4\pi$. The potential of a simple layer is continuous, but the potential of a double layer jumps by a finite amount in passing across the surface, such that



$$W_i = W_S + 2\pi\nu_S, \qquad W_e = W_S - 2\pi\nu_S;$$

therefore

$$W_e - W_i = -4\pi\nu_S.$$

Now we consider the normal derivative. Take a point Q on S and let the inner normal be n_Q; then, take a point P on n_Q. We have

$$\frac{\partial V}{\partial n_Q} = \int\int_S \mu \frac{\partial\left(\frac{1}{r}\right)}{\partial n_Q} d\sigma \qquad \text{from} \qquad V = \int\int_S \frac{\mu}{r} d\sigma.$$

Furthermore, because

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n_Q} = -\frac{1}{r^2}\frac{\partial r}{\partial n_Q} = -\frac{\cos (r, n_Q)}{r^2},$$

we have

$$\frac{\partial V}{\partial n_Q} = -\int\int_S \frac{\mu \cos (r, n_Q)}{r^2} d\sigma.$$

Denote the variable normal in the integration over S by n_R, then

$$\frac{\partial V}{\partial n_Q} = -\int\int_S \frac{\mu \cos (r, n_R)}{r^2} d\sigma + \int\int_S \mu \frac{\cos (r, n_R) - \cos (r, n_Q)}{r^2} d\sigma = W + C.$$

We can prove that the second integral C remains unaltered in passing across S. (See, for example, Kron, 1899; Bouligand, 1926; Sternberg, 1925b; Poincaré, 1899; Kellogg, 1929; Gunther, 1934.)

$$\frac{\partial V_i}{\partial n_Q} = -2\pi\ \mu_Q + \epsilon, \qquad \frac{\partial V_e}{\partial n_Q} = 2\pi\ \mu_Q + \epsilon,$$

$$\epsilon = -\int\int_S \frac{\mu \cos\,(r_{RQ}, n_Q)}{r_{RQ}{}^2} d\sigma$$

$$= \int\int_S \mu \frac{\partial}{\partial n_Q}\left(\frac{1}{r}\right) d\sigma,$$

or

$$\frac{1}{2}\left(\frac{\partial V_i}{\partial n_Q} - \frac{\partial V_e}{\partial n_Q}\right) = -2\pi\ \mu_Q,$$

$$\frac{1}{2}\left(\frac{\partial V_i}{\partial n_Q} + \frac{\partial V_e}{\partial n_Q}\right) = \epsilon.$$

If

$$V_i = \int\int_S \frac{\mu}{r} d\sigma,$$

then inside S,

$$\lim \frac{\partial}{\partial n_Q} \int\int_S \frac{\mu}{r} d\sigma = -2\pi\mu_Q + \int\int_S \mu_R \frac{\cos\,(n_R, r)}{r^2} d\sigma; \tag{7}$$

outside S,

$$\lim \frac{\partial}{\partial n_Q} \int\int_S \frac{\mu}{r} d\sigma = 2\pi\mu_Q + \int\int_S \mu_R \frac{\cos\,(n_R, r)}{r^2} d\sigma. \tag{7a}$$

Take a Newtonian potential

$$V = \sum \frac{m_i}{r_i},$$

or

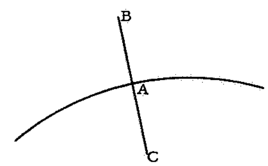
$$V = \int\int\int \frac{\rho' d\tau'}{r}$$

if the volume distribution is continuous with volume density ρ, or

$$V = \int\int \frac{\mu' d\sigma'}{r}$$

if the surface distribution is continuous with surface density μ. Potential V is continuous with its first derivative in the whole space and satisfies $\Delta V = 0$ outside the attracting mass, and satisfies $\Delta V = -4\pi\rho$ at a point of density ρ. The second derivative is discontinuous on the boundary of two different media. The derivative of V is discontinuous on the surface itself. Take two lengths $AB = dn_e$ and $AC = dn_i$ on the normal at a point A on S;

$$V_B=V_A+\frac{\partial V}{\partial n_e}\,dn_e,\qquad V_C=V_A+\frac{\partial V}{\partial n_i}\,dn_i.$$



On the other hand,

$$\frac{\partial V}{\partial n_\epsilon}+\frac{\partial V}{\partial n_i}=-4\pi\mu,$$

where μ is the density of the attracting mass. The tangential derivative, however, is continuous. If the layer is double, then

$$W=-\int\int_S \nu\frac{\partial}{\partial n_e}\left(\frac{1}{r}\right)d\sigma'=-\int\int_S \frac{\nu\cos(r,n_e)}{r^2}\,d\sigma'.$$

Hence we obtain Liapounov's relation

$$\frac{\partial W}{\partial n_e}+\frac{\partial W}{\partial n_i}=0,$$

$$W_e-W_i=-4\pi\nu,\qquad W_e+W_i=-2\int_S \nu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\sigma.$$

In this case, the tangential derivative is discontinuous. Hence any function that is harmonic and continuously differentiable in a closed regular domain $T+S$ can be represented as the sum of the potentials of a simple and a double layer on S.

Dirichlet's Problem

It is required to obtain a function V such that

(1) It is regular at all points of T and continuous with its normal derivative as we approach the boundary S,

(2) It satisfies $\Delta V=0$ at all points of T, and

(3) It reduces to a given function F on S.

This is called the *Dirichlet interior problem*. When we consider instead of T the space outside S extending to infinity and add a further condition that V vanishes at infinity, then this is called the *Dirichlet exterior problem*. In these cases, the function F is given on S; this is the first boundary value problem. If, instead of the given function V on S, $\partial V/\partial n$ should reduce to a given function F on S, then this is the second boundary value problem. If $k(\partial V/\partial n)+hV=F$ is given on S, then this is the third boundary value problem, where k, h, and F are continuous and k and h are positive. The second boundary value problem of Dirichlet is sometimes called the *Neumann problem* (Poincaré, 1899; Gunther, 1934).

The solution of the first boundary value problem inside a sphere is given by

$$u(x,y,z)=\frac{1}{2\pi}\int\int_S F\left[\frac{\cos(r,n)}{r^2}-\frac{1}{2hr}\right]d\sigma,$$

where h is the radius of the circle. In fact, the first integral is the potential W of a double layer, and the second that of a simple layer; both integrals are regular inside the sphere. When $P(x,y,z)$ approaches S, the potential W takes the form

$$W_i = F_S + W_S = F_S + \frac{1}{2\pi} \int \int_S F \frac{\cos(r_{SQ}, n_Q)}{r_{SQ}^2} d\sigma.$$

Since $\cos(r, n) = r/2h$ on S, we have

$$W_i = F_S + \frac{1}{4\pi h} \int \int_S \frac{F}{r_{SQ}} d\sigma.$$

But

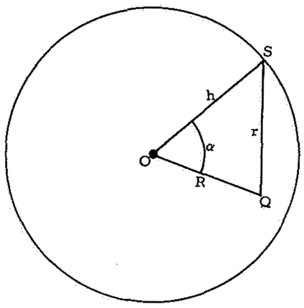
$$V_i = V_S = \frac{1}{4\pi h} \int \int_S \frac{F}{r_{SQ}} d\sigma.$$

Hence,

$$U_i = W_i - V_i = F_S.$$

Let $R^2 = h^2 + r^2 - 2hr \cos(r, n)$; then

$$\frac{\cos(r, n)}{r^2} - \frac{1}{2hr} = \frac{h^2 - R^2}{2hr^3},$$



and we have

$$u(x, y, z) = \frac{1}{4\pi h} \int \int_S F \frac{h^2 - R^2}{(h^2 + R^2 - 2hR \cos \alpha)^{3/2}} d\sigma.$$

This is called the *Poisson integral.*

The *Green function* G is a function of two points $P(x, y, z)$ and $Q(\xi, \eta, \zeta)$, where P varies inside T and Q varies inside T and on S. Function G is regular in T except at P as a function of ξ, η, ζ, and represents continuous potential in $T+S$. It becomes infinite in the order of the reciprocal of $\overline{PQ}$ at point P, such that

$$G(x, y, z; \xi, \eta, \zeta) - \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} = W(x, y, z; \xi, \eta, \zeta)$$

is a regular potential at P, and vanishes at all points Q of S and at all points P of T. To determine function G is one of the first boundary value problems because W takes the known value $-1/r$ on the boundary S. Let u be regular in T and be continuous with its first normal derivative as we approach S, then from equations 6a and 2a

$$u = -\frac{1}{4\pi} \int \int_S u \frac{\partial G}{\partial n} d\sigma,$$

where (6b)

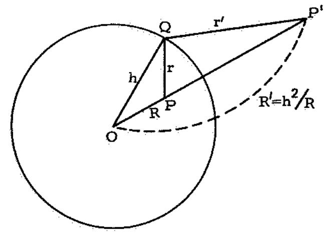
$$G = 1/r + V.$$

The Green function has a reciprocity relation

$$G(x, y, z; \xi, \eta, \zeta) = G(\xi, \eta, \zeta; x, y, z),$$

and

$$G = \frac{1}{r} - \frac{h}{R}\frac{1}{r'}.$$



inside a sphere.

Physically speaking, V is the potential of the induced charge on the conductor S by a unit charge at P, and the Green function G is the value at Q of the potential due to the induced charge and the charge at P. This circumstance was the basis of the proof of the existence of the Green function by Green, Kirchhoff, and Thomson. This was challenged by Lebesgue. The rigorous mathematical proof of the existence of harmonic functions was discussed by Gauss, Riemann, Dirichlet, Weierstrass, Hilbert, Schwarz, Neumann, Hadamard, and Poincaré, in connection with the existence of analytic functions on the Riemann surface (Courant, 1950; Weyl, 1923). Recent development—in particular, the theory of capacity—deals with boundaries with discontinuous normals on the point-set theoretical ground (Kellogg, 1929).

Consider the first interior boundary value problem. The required function is given on S such that $u_i = F$. The potential of a double layer is

$$u = \int\int_S \frac{\nu \cos(r, n)}{r^2}\, d\sigma,$$

and we have

$$u_i = 2\pi\nu(S) + \int\int_S \nu(\sigma) \frac{\cos(r_{\sigma S}, n_\sigma)}{r^2_{\sigma S}}\, d\sigma = F(S).$$

where $r_{\sigma S}$ denotes the distance between σ and S. Let

$$K(S, \sigma) = \frac{1}{2\pi} \frac{\cos(r_{\sigma S}, n_\sigma)}{r^2_{\sigma S}};$$

then

$$\nu(S) - \lambda \int\int_S K(S, \sigma)\nu(\sigma)\, d\sigma = \frac{F(S)}{2\pi}.$$

The kernel $K(S, \sigma)$ becomes infinite in the order of $1/r_{\sigma S}$ as $\sigma \to S$, because

$$\frac{\cos(r_{\sigma S}, n_\sigma)}{r_{\sigma S}}$$

is bounded on S. In this case, the iterated kernel

$$K^3(S, \sigma) = \int\int\int\int K(S\sigma_1)K(\sigma_1\sigma_2)K(\sigma_2\sigma)\, d\sigma_1\, d\sigma_2$$

is shown to be bounded on S even when $\sigma \to S$. If the moment $\nu(S)$ of the double layer is known by solving the integral equation, then the potential u is known.

For the second boundary value problem, we are given

$$\frac{\partial u_i}{\partial n}=F.$$

Take

$$u=\int\int_S \frac{\mu}{r}\,d\sigma,$$

then

$$\frac{\partial u_i}{\partial n}=2\pi\mu(S)+\int\int_S \mu(\sigma)\,\frac{\cos\,(r,\,n_S)}{r^2}\,d\sigma=F(S).$$

For the third boundary value problem we are given

$$\frac{\partial u_i}{\partial n}+hu_i=F.$$

Let

$$u=\int\int_S \frac{\mu}{r}\,d\sigma;$$

then

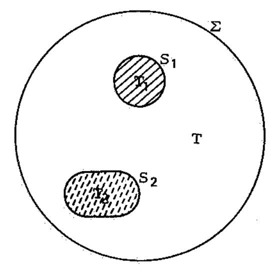
$$\frac{\partial u_i}{\partial n}+hu_i=-2\pi\mu(S)+\int\int_S \mu(\alpha)\left[\frac{\cos\,(r,\,n_S)}{r^2}+\frac{h(\sigma)}{r}\right]d\sigma=F(S).$$

Thus the boundary value problem is reduced to the solution of an integral equation of Fredholm's type (Plemelj, 1911; Neumann, 1905, 1912; Kneser, 1922; Wangerin 1922; Kellogg 1929).

Poincaré's Formula

Let V_1, V_2 be functions of the nature of potentials; V_1 depends on ρ_1 in T_1+S_1, and V_2 depends on ρ_2 in T_2+S_2. In the whole space,

$$\oint V_2\Delta V_1 d\tau+\oint\left(\frac{\partial V_2}{\partial x}\frac{\partial V_1}{\partial x}+\ .\ .\ .\right)d\tau=0.$$



In fact, we draw a large sphere Σ with the coordinate origin as center and large enough to include both T_1 and T_2. In the space T between S_1 and Σ, equation 1 reduces to

$$\int_T V_2\Delta V_1 d\tau+\int_T\left(\frac{\partial V_1}{\partial x}\frac{\partial V_2}{\partial x}+\ .\ .\ .\right)d\tau=\int_\Sigma V_2\frac{\partial V_1}{\partial n}\,d\sigma+\int_{S_1} V_2\frac{\partial V_1}{\partial n_i}\,d\sigma.$$

As we increase the radius, V_2 is of order $1/R$ on S and

$$\frac{\partial V_1}{\partial n}=\frac{\partial V_1}{\partial R}$$

is of order $1/R^2$, and $d\sigma=R^2d\omega$, where $d\omega$ is the element of a solid angle. Hence

$$\int V_2 \frac{\partial V_1}{\partial n} d\sigma \quad \text{is of order } 1/R.$$

Next apply Green's formula in S_1+T_1:

$$\int_{T_1} V_2 \, \Delta V_1 \, d\tau + \int_{T_1} \left(\frac{\partial V_2}{\partial x} \frac{\partial V_1}{\partial x} + . \, . \, . \right) d\tau = \int_{S_1} V_2 \frac{\partial V_1}{\partial n_e} d\sigma.$$

On S, we have

$$\frac{\partial V}{\partial n_i} = -\frac{\partial T}{\partial n_e}.$$

Hence by adding these two we obtain

$$\oint V_2 \, \Delta V_1 \, d\tau + \oint \left(\frac{\partial V_2}{\partial x} \frac{\partial V_1}{\partial x} + . \, . \, . \right) d\tau = 0. \tag{8}$$

By exchanging V_1 and V_2, and by subtracting the result, we get

$$\int (V_2 \, \Delta V_1 - V_1 \, \Delta V_2) \, d\tau = 0, \qquad \text{or} \qquad \int (\rho_1 V_2 - \rho_2 V_1) \, d\tau = 0.$$

$\rho_1 = 0$ outside T_1; $\rho_2 = 0$ outside T_2. Hence

$$\int_{T_1} \rho_1 V_2 \, d\tau = \int_{T_2} \rho_2 V_1 \, d\tau. \tag{9}$$

This formula is analogous to the case of electricity in which $\sum m_1 V_2 - \sum m_2 V_1 = 0$. If we let $V_2 = V_1 + dV_1$, $\rho_2 = \rho_1 + d\rho_1$, then

$$\int (V_1 d\rho_1 - \rho_1 dV_1) \, d\tau = 0. \tag{10}$$

Suppose that a system of masses m', m'', . . . attracts another system of masses m_1', m_1'', ... Let V_1', V_1'', . . . be the potentials at the points m', m'', . . ., due to all the attracting masses m_1', m_1'', . . ., and V', V'', . . . be the potentials at points m_1', m_1'', . . . due to all the attracting masses m', m'', ... If the attracted mass is displaced, then the work done is

$$\epsilon = \sum m \left(\frac{\partial V_1}{\partial x} \delta x + \, . \, . \, . \right) = \sum m \delta V_1.$$

Let $H = \Sigma m V_1$; then $\epsilon = \delta H$. If the attracting mass is displaced, then V_1 changes to $V_1 + \delta' V_1$ and $\delta' H = \Sigma m_1 \delta' V_1'$. If these two displacements occur simultaneously, then $\epsilon = \delta H + \delta' H$. Denote the volume of the attracted masses by T; then

$$H = \int_T \rho V_1 d\tau.$$

Since $\rho=0$ outside T, we can extend this integral to the whole space. We have $\delta H=\oint \rho\delta V_1 d\tau$. If the attracting mass is the same as the attracted mass, then $V=V_1$ and $\delta H=\int \rho\delta V d\tau$. By equation 10 we have

$$\delta H=\int \rho\delta V d\tau=\int \frac{\rho\delta V+V\delta\rho}{2}\, d\tau.$$

Let $W=\oint(\rho V/2)d\tau$. This is the energy of the system. We have $\delta W=\epsilon$, but $\Delta V=-4\pi\rho$. Hence

$$W=-\frac{1}{8\pi}\int V\Delta d\tau.$$

From Poincaré's formula, equation 8, we get finally

$$W=\frac{1}{8\pi}\oint\left[\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right]d\tau. \tag{11}$$

GENERAL PROPERTIES OF EQUILIBRIUM FIGURES

Hydrostatic Equilibrium

Consider a mass of incompressible fluid rotating around a fixed axis without any external force. Take the rotating axis as the z-axis and assume that the angular velocity of the rotation is a constant ω. The x- and y-axes are fixed in the fluid mass, and the center of mass is on the z-axis. Take the center of mass as origin. Then denote by p the pressure at a point (x, y, z) of the fluid. The pressure p depends only on x, y, z. The force acting on a volume element of the fluid is

$$Xd\tau=\frac{\partial p}{\partial x}d\tau, \qquad Yd\tau=\frac{\partial p}{\partial y}d\tau, \qquad Zd\tau=\frac{\partial p}{\partial z}d\tau,$$

where X, Y, Z are the components of force acting on the molecule at the point (x, y, z). Then

$$X=\rho\frac{\partial V}{\partial x}+\omega^2\rho x; \qquad Y=\rho\frac{\partial V}{\partial y}+\omega^2\rho y; \qquad Z=\rho\frac{\partial V}{\partial z}.$$

We obtain the condition of relative equilibrium, by writing $U=V+(\omega^2/2)(x^2+Y^2)$, *as*

$$\frac{\partial p}{\partial x}=\rho\frac{\partial U}{\partial x}; \qquad \frac{\partial p}{\partial y}=\rho\frac{\partial U}{\partial y}; \qquad \frac{\partial p}{\partial z}=\rho\frac{\partial U}{\partial z}.$$

From these we have

$$\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz=\rho\left(\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy+\frac{\partial U}{\partial z}dz\right).$$

Hence,

$$dp=\rho dU.$$

If $du=0$, then $dp=0$, and therefore p is a function of U only. The density ρ is also a function of U only. The surface $U=$ constant is called the "level surface." Consequently the level surface is the surface of equal pressure and also the surface of equal density.

On the surface S the pressure $p=0$; hence U is constant on S, and the free surface is accordingly a level surface. If there is no rotation, then $U=V$. In general $\Delta V=-4\pi\rho$; this ΔV is a function of U. If the surface consists of several pieces of surfaces, then on each of the surfaces S_i, we have

$$U=V+\frac{\omega^2}{2}(x^2+y^2) \qquad \text{is a constant;}$$

that is, the equation for the surface is $U=$constant. U has the property of a potential, and the gravity

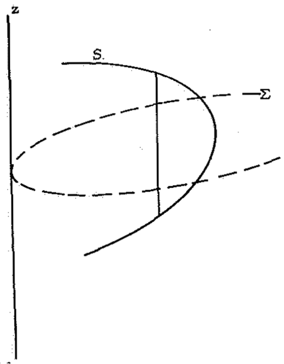
$$\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]^{1/2}$$

is zero nowhere on the surface. Thus $\partial U/\partial x$, $\partial U/\partial y$, $\partial U/\partial z$ cannot vanish all at once. Suppose that $\partial V/\partial z\neq 0$. Then $U=V+(\omega^2/2)(x^2+y^2)$ can be solved for z, and S is a regular surface. The surfaces $U=$constant, $\rho=$constant, $p=$constant coincide and become what is called the "equipotential surface." The force is directed normally to the equipotential surface.

Symmetry Plane

Theorem: $z=0$ is always a symmetry plane of the body T.

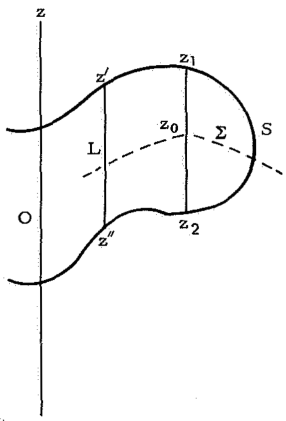
For proof, we take the locus Σ of the middle point of the chord parallel to the rotation-axis. When the chord intersects at more than two points with S, we take the middle point of the chord inside S. If Σ is not a plane, then there is at least one point $Q(x_0, y_0, z_0)$ inside T or on S, such that this z_0 is the upper bound of all values of z of Σ and there is at least one point Q on Σ whose z is smaller than z_0. At first suppose that Q is inside of T. The straight line $x=x_0$, $y=y_0$ intersects S in a finite or an infinite number of points. Let $P_1(x_0, y_0, z_1)$, $P_2(x_0, y_0, z_2)$ be such points which are the nearest to $Q(x_0, y_0, z_0)$. Suppose $z_1>z_2$ then $[V(x, y, z)]+(\omega^2/2)(z^2+y^2)$ is constant in each component of S. Hence



$$V(x_0, y_0, z_1)=V(x_0, y_0, z_2). \tag{12}$$

Denote the projection of S on the plane $z=0$ by D. Then

$$V(\bar{x}, \bar{y}, \bar{z})=\int_D dxdy\int_L \frac{1}{r}dz, \tag{13}$$



where L is a straight line parallel to Oz inside of T. Let the intersection of L with S be z', z''; and $r(z, z_1)$ the distance of the points (x, y, z) and (x_0, y_0, z_1); and $r(z, z_2)$ the distance of the points (x, y, z) and (x_0, y_0, z_2). The centers of mass of two chords $(z'z'')$ and (z_1z_2) have a common z-coordinate. In fact, by our assumption, $r(z, z_1)>r(z, z_2)$ for every z. Hence

$$\int_{z''}^{z'} \frac{1}{r(z, z_1)}\, dz \leqslant \int_{z''}^{z'} \frac{1}{r(z, z_2)}\, dz. \tag{14}$$

From equations 13 and 14 we should have

$$V(x_0, y_0, z_1) < V(x_0, y_0, z_2). \tag{15}$$

This contradicts equation 12.

We must next consider the case when the upper bound of the z-coordinates of all points of Σ is attained at a point (x_0, y_0, z_0) on S. If the gravity at (x_0, y_0, z_0) is zero, then

$$(\partial/\partial z)U(x_0, y_0, z_0) = 0; \qquad \text{thus} \qquad (\partial/\partial z)V(x_0, y_0, z_0) = 0.$$

If the gravity at (x_0, y_0, z_0) is not zero, then S has a continuous normal both at (x_0, y_0, z_0) and near (x_0, y_0, z_0). Then the straight line $x=x_0$, $y=y_0$ must touch S at (x_0, y_0, z_0). Hence in this case, too, we should have $(\partial/\partial z)U=0$ and $(\partial/\partial z)V=0$. This relation is impossible.

It is permissible in the case of a cylinder that V is independent of z. A cylinder has a symmetry plane perpendicular to Oz; hence Σ is one plane region. Accordingly S has a symmetry plane perpendicular to the z-axis. As we have taken the center of mass of T as the origin, this is the (xy)-plane. Since the rotation axis is perpendicular to the symmetry plane and passes through the center of mass, this is one of the principal axes of the body T.

As a corollary, there are only two points in which the straight line parallel to the rotation axis meets the surface S.

When there are several components of S, they are not situated in the direction parallel to the z-axis, but in the direction perpendicular to it. If several components have a common point, the point should be on the symmetry plane. A straight line passing through this point and parallel to the rotation axis does not intersect any component of S. Each component mass of the fluid consists of only one boundary continuum; it has no hollow part (Lichtenstein, 1928, 1918).

Gravity

Theorem: Any point where gravity vanishes lies on the plane $z=0$.

Take a point (x', y', z') with $z'>0$ on S. Denote the part of T which is above z' by Θ', and its image with regard to $z=z'$ by $\bar{\Theta}'$. The component along the straight line $x=x'$, $y=y'$ of the attracting force of $\Theta'+\bar{\Theta}'$ at (x', y', z') is zero by symmetry. The component along that direction of the force due to $T-\Theta'-\bar{\Theta}'$ is certainly negative. Consider that $(\partial/\partial z)V(x', y', z')<0$; hence $(\partial/\partial z)U(x', y', z')<0$. The equality must occur on $z=0$.

Theorem: Gravity is directed everywhere inward. If T consists of several bodies and each of the components has continuous normals, then the bodies cannot have common points but must be separated. The gravity at a common point is zero, and the common point is a point of discontinuity.

Let a point of S be $P(x, y, z)$. The Green function $G(\bar{x}, \bar{y}, \bar{z}; x, y, z)$ has continuous partial derivatives of the first order on S as a function of x, y, z. The derivative $(\partial/\partial n)G(\bar{x}, \bar{y}, \bar{z}; x, y, z)$ is continuous, and $G(\bar{x}, \bar{y}, \bar{z}; x, y, z)>0$ for a point $(\bar{x}, \bar{y}, \bar{z})$ inside T. Hence $(\partial/\partial n_i)G(\bar{x}, \bar{y}, \bar{z}; P)\geqslant 0$; I shall prove that only the inequality holds.

Take a point $(\underline{x}, \underline{y}, \underline{z})$ on the normal, very near point P. We can write $G(\underline{x}, \underline{y}, \underline{z}; x, y, z) = 1/\underline{r} + g(\underline{x}, \underline{y}, \underline{z}; x, y, z)$, where $\underline{r}$ is the distance between $(\underline{x}, \underline{y}, \underline{z})$ and (x, y, z). Gravity g is continuous inside T and on S, and regular in T, and takes the value $-[(\underline{x}-x')^2+(\underline{y}-y')^2+(\underline{z}-z')^2]^{-1/2}$ at (x', y', z') on S. This is negative and takes the minimum value at P.

Hence $(\partial/\partial n_i)g(\underline{x}, \underline{y}, \underline{z}; P) \geqslant 0$. The normal derivative of $1/\underline{r}$ at P is positive. Therefore $(\partial/\partial n_i)G(\underline{x}, \underline{y}, \underline{z}; P) > 0$.

We can prove that $(\partial/\partial n_i)G(\bar{x}, \bar{y}, \bar{z}; P) > 0$ at any point $(\bar{x}, \bar{y}, \bar{z})$ inside T. In fact, $(\partial/\partial n_i)G(\bar{x}, \bar{y}, \bar{z}; P)$ is a regular potential function if P is fixed. This cannot have a minimum inside T. The derivative $\partial G/\partial n_i$ is equal to 0 on S and $\geqslant 0$ inside T. Hence inside T it should be either always positive or always negative. At a point $(\underline{x}, \underline{y}, \underline{z})$ we have shown that $\partial G/\partial n_i > 0$. Thus it should be always $\partial G/\partial n_i > 0$ inside T.

By the reciprocity relation of a Green function, we have $G(x, y, z; \bar{x}, \bar{y}, \bar{z}) = G(\bar{x}, \bar{y}, \bar{z}; x, y, z)$, and hence $(\partial/\partial n_i)G(P; \bar{x}, \bar{y}, \bar{z}) > 0$. By writing $\Delta U = -4\pi\rho + 2\omega^2$ after differentiating equation 17 which will be proved immediately we obtain

$$\frac{\partial U}{\partial n_i} = -\frac{1}{4\pi}\int_T \frac{\partial}{\partial n_i} G(P; x, y, z)(2\omega^2 - 4\pi\rho)\,d\tau.$$

Since $(\partial/\partial n_i)G(P; x, y, z) > 0$, $\partial U/\partial n_i > 0$, if $2\omega^2 < 4\pi\rho$.

Since the gravity should be directed inward at the contact point, it should be zero. Hence, the curvature should have a discontinuity at the contact point; that is, there is a conical point as in Darwin's conjectural double-star model (1906, 1910).

Angular Velocity

A sufficient (though not necessary) condition for equilibrium is that the force at every point of the free surface be directed inward. Otherwise the equilibrium would break down. Hence

$$\frac{\partial U}{\partial n_e} < 0;$$

therefore

$$\int \frac{\partial U}{\partial n_e}\,d\sigma < 0.$$

From equation 3 we have

$$\int_T \Delta U\,d\tau < 0.$$

Therefore

$$\int_T \Delta V\,d\tau + \int_T 2\omega^2\,d\tau < 0.$$

Hence,

$$-4\pi\int_T \rho\,d\tau + \int_T 2\omega^2\,d\tau < 0.$$

Let M be the mass, and T be the volume. Then $-4\pi M + 2\omega^2 T < 0$. If the density is ρ, then

$$2\omega^2 < 4\pi\rho. \tag{16}$$

This is called Poincaré's inequality (1885). The inequality $\partial U/\partial n_i > 0$ has been proved under the assumption $2\omega^2 < 4\pi\rho$, and the latter inequality has been proved by Poincaré under the assumption $\partial U/\partial n_i > 0$.

We shall examine whether the gravity becomes zero at some points although the proof is unnecessary if S is continuously curved.

(I) $U(x, y, z) = V(x, y, z) + (\omega^2/2)(x^2+y^2)$ should have a constant value on each of the components.

(II) If the fluid cannot stand any tension (Zugspannung), the gravity should be directed inward or be zero on the surface S.

Proposition II follows as a necessary condition from Proposition I. If Proposition I is satisfied, the pressure exists in the whole T, and equilibrium takes place; that is, it is a sufficient condition to achieve equilibrium. Let us proceed to the proof according to Lichtenstein (1918).

If $\omega^2 > 2\pi\rho$, then Proposition I is not satisfied; $U = V + (\omega^2/2)(x^2+y^2)$ takes the value U_0 on S, and $\Delta U = -4\pi\rho + 2\omega^2$ inside T. Take a point $(\bar{x}, \bar{y}, \bar{z})$ inside T and let the Green function at $(\bar{x}, \bar{y}, \bar{z})$ inside T with respect to T, which vanishes on S, be $G_0(\bar{x}, \bar{y}, \bar{z}; x, y, z)$. Obviously $G_0(\bar{x}, \bar{y}, \bar{z}; x, y, z) > 0$ for all pairs of points $(\bar{x}, \bar{y}, \bar{z})$ and (x, y, z) inside T. We then have

$$U(\bar{x}, \bar{y}, \bar{z}) = U_0 - \frac{1}{4\pi}\int_T G_0(\bar{x}, \bar{y}, \bar{z}; x, y, z)(2\omega^2 - 4\pi\rho)\,d\tau. \tag{17}$$

Let W assume a value such that $\Delta W = 0$, and let $W + 1/r = G$. From equation 2,

$$-\int W\Delta U d\tau = \int \left(U\frac{\partial W}{\partial n} - W\frac{\partial U}{\partial n}\right) d\sigma.$$

From equation 6 we have

$$4\pi U = -\int \left(U\frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n}\right) d\sigma - \int \frac{\Delta U}{r} d\tau.$$

Hence,

$$4\pi U = -\int U\frac{\partial G}{\partial n}\,d\sigma - \int G\Delta U d\tau.$$

But from equation 6b,

$$4\pi U_0 = -\int U\frac{\partial G}{\partial n}\,d\sigma.$$

Accordingly,

$$U = U_0 - \frac{1}{4\pi}\int G_0 \Delta U d\tau.$$

If we suppose $\omega^2 < 2\pi\rho$, then by equation 17, $U(\bar{x}, \bar{y}, \bar{z}) > U_0$ at every point (x, y, z) inside T. Since the external pressure is zero on the boundary, pressure always exists inside T because of this inequality. Hence, the proposition has been proved.

If $\omega^2 \geqslant \pi\rho$, then an equilibrium figure cannot exist for a convex body. The proof is based on the extremum property discussed in Blaschke (1916).

Crudeli (1909, 1910) obtained a sharper limit than the one in Poincaré's inequality.

We have at first $U = V + (\omega^2/2)\ (x^2+y^2) = C, \partial U/\partial n < 0$, and $\Delta U =$ constant inside T. From equation 6

$$4\pi U = -\int_S \left(U \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n} \right) d\sigma - \int_T \frac{\Delta U}{r} d\tau.$$

Substituting $\Delta U = \Delta V + 2\omega^2 = 2\omega^2 - 4\pi\rho$ (if $\rho =$ const.) gives

$$4\pi U = 4\pi C + \int_S \frac{1}{r}\frac{\partial U}{\partial n} d\sigma - \frac{2\omega^2 - 4\pi\rho}{\rho} V, \tag{18}$$

because $U = C$ on the free surface S, and

$$\int_S \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\sigma = -4\pi, \qquad V = \rho \int_T \frac{d\tau}{r}.$$

Denoting the inner normal at a point A of an equipotential surface by n_i and differentiating the above equation give

$$4\pi \frac{\partial U}{\partial n_i} = \frac{\partial}{\partial n_i} \int_S \frac{1}{r}\frac{\partial U}{\partial n} d\sigma - \frac{(2\omega^2 - 4\pi\rho)}{\rho}\frac{\partial V}{\partial n_i},$$

where r is the distance between an inside point P on the normal n_i and the point A. At the point A, this expression is

$$4\pi \left(\frac{\partial U}{\partial n_i}\right)_A = \lim \frac{\partial}{\partial n_i} \int_S \frac{1}{r}\frac{\partial U}{\partial n} d\sigma - \frac{2\omega^2 - 4\pi\rho}{\rho}\left(\frac{\partial V}{\partial n_i}\right)_A.$$

But from equation 17 where we write $-\mu = \partial U/\partial n$, we have

$$\lim \frac{\partial}{\partial n_i} \int_S \frac{1}{r}\frac{\partial U}{\partial n} d\sigma = 2\pi \left(\frac{\partial U}{\partial n_i}\right)_A + \int_S \frac{\partial U}{\partial n}\frac{\cos(r_0, n_i)}{r_0^2} d\sigma,$$

where $\overline{AS} = r_0$. Hence, we obtain

$$2\pi \left(\frac{\partial U}{\partial n_i}\right)_A - \int_S \frac{\partial U}{\partial n}\frac{\cos(r_0, n_i)}{r_0^2} d\sigma = -\frac{2\omega^2 - 4\pi\rho}{\rho}\left(\frac{\partial V}{\partial n_i}\right)_A.$$

By the relation

$$\frac{\partial V}{\partial n_i} = \frac{\partial U}{\partial n_i} - \frac{\omega^2}{2}\frac{\partial (x^2+y^2)}{\partial n_i}$$

we have

$$2(\omega^2 - \pi\rho)\left(\frac{\partial U}{\partial n_i}\right)_A - \rho \int_S \frac{\partial U}{\partial n}\frac{\cos(r_0, n_i)}{r_0^2} d\sigma = \omega^2(\omega^2 - 2\pi\rho)\left[\frac{\partial (x^2+y^2)}{\partial n_i}\right]_A.$$

Let $A \to S$; if A is the point where the tangent plane to S is perpendicular to the rotation axis, that is, where

$$\left[\frac{\partial(x^2+y^2)}{\partial n_i}\right]_A=0,$$

then at that point A

$$2(\omega^2-\pi\rho)\left(\frac{\partial U}{\partial n_i}\right)_A-\rho\int_S\frac{\partial U}{\partial n}\frac{\cos(r_o,n_i)}{r_0^2}d\sigma=0.$$

Where,

$$\cos(r_0,n_i)>0,\qquad \partial U/\partial n<0.$$

Hence,

$$\rho\int_S\frac{\partial U}{\partial n}\frac{\cos(r_o,n_i)}{r_0^2}d\sigma<0.$$

Consequently

$$(\omega^2-\pi\rho)\left(\frac{\partial U}{\partial n_i}\right)_A<0,$$

and

$$\omega^2-\pi\rho<0.$$

Thus we get Crudeli's inequality

$$\omega^2<\pi\rho. \tag{19}$$

As a corollary, this relation holds true for an elliptic cylinder (Tisserand, 1891, p. 107), because, from $\omega^2=4\pi\rho[ab/(a+b)^2]$, we get

$$\frac{a-b}{a+b}=\sqrt{1-\frac{\omega^2}{\pi\rho}}.$$

By computing gravity, von Zeipel (1898) showed that

$$\frac{\omega^2}{2\pi\rho}<\frac{4}{9},$$

because gravity on the equator

$$g_a=-4\pi\rho a\left[1-\frac{\omega^2}{2\pi\rho}\frac{3\arctan\epsilon-\epsilon(3-\epsilon^2)}{(3+\epsilon^2)\arctan\epsilon-3\epsilon}\right]<0.$$

Nikliborc (1929) showed that Crudeli's inequality holds even when the equilibrium figure consists of a finite number of bounded domains.

Ellipticity

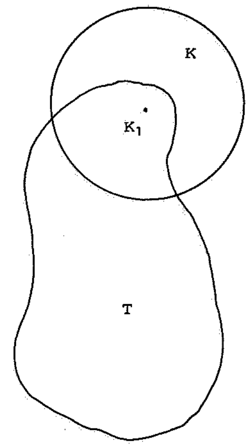
We first prove that

$$\int_T\frac{1}{r^2}d\tau\leqslant 4\pi\left(\frac{3T}{4\pi}\right)^{1/3}.$$

The integral on the left-hand side taken over the volume T of the sphere drawn with the center at the point in question is equal to the right-hand side. Let the radius of the sphere be R and the domain common to K and T be K_1. Then,

$$4\pi\left(\frac{3T}{4\pi}\right)^{1/3}=\int_K \frac{1}{r^2}\,d\tau=\int_{K_1}\frac{1}{r^2}\,d\tau+\int_{K-K_1}\frac{1}{r^2}\,d\tau$$

$$\geqslant\int_{K_1}\frac{1}{r^2}\,d\tau+\frac{1}{R^2}\int_{K-K_1}d\tau=\int_{K_1}\frac{1}{r^2}\,d\tau+\frac{T-K_1}{R^2};$$



$$\int_T\frac{1}{r^2}\,d\tau=\int_{K_1}\frac{1}{r^2}\,d\tau+\int_{T-K_1}\frac{1}{r^2}\,d\tau$$

$$\leqslant\int_{K_1}\frac{1}{r^2}\,d\tau+\frac{1}{R^2}\int_{T-K_1}d\tau=\int_{K_1}\frac{1}{r^2}\,d\tau+\frac{T-K_1}{R^2}.$$

Combining these two equations provides the formula we are to prove. Since

$$\frac{\partial V}{\partial x}=\int_T\frac{\partial}{\partial x}\frac{1}{r}\,dxdydz,$$

we have

$$\left|\frac{\partial V}{\partial x}\right|<\int_T\frac{1}{r^2}\,dxdydz<\frac{4\pi}{3}\left(\frac{3T}{4\pi}\right)^{1/3}.$$

Similarly,

$$\left|\frac{\partial V}{\partial y}\right|,\qquad\left|\frac{\partial V}{\partial z}\right|<\frac{4\pi}{3}\left(\frac{3T}{4\pi}\right)^{1/3}.$$

Hence

$$\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2}<\frac{4\pi}{\sqrt{3}}\left(\frac{3T}{4\pi}\right)^{2/3}.$$

Now let B be a point on S and assume that the gravity reaches its maximum at B; B is obviously on the symmetry plane. The gravity at B is either directed inward or equal to zero. The centrifugal force at B cannot be larger than the attracting force. Hence

$$\omega^2 a\leqslant\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]^{1/2}<\frac{4\pi}{\sqrt{3}}\rho\left(\frac{3T}{4\pi}\right)^{1/3}.$$

Hence,

$$a<\frac{4\pi}{\sqrt{3}}\frac{\rho}{\omega^2}\left(\frac{3T}{4\pi}\right)^{1/3}=a_0. \tag{20}$$

Accordingly, the equilibrium figure is inside the circular cylinder of radius a_0 (Schmidt, 1914; Lichtenstein, 1918). Mazurkiewicz (1926) proved that if $b>a$, then

$$b<CT^{1/3},\qquad C=e^{168050},$$

where a is the distance from the rotation-axis and $2b$ the thickness of the figure in the z-direction.

Nikliborc (1931, 1933) proved that if the meridian section of the figure of the spherical type is monotonous for $z>0$, then the ellipticity $b/a<10$ and furthermore, that for a figure for which the highest point lies on the z-axis, $b/a<5$. For a plane figure, Nikliborc (1932) and Blaschke (1932) proved similar theorems.

Merlin (1935) proved that

$$a<\frac{4\pi\rho}{\omega^2}\left(\frac{3T}{4\pi}\right)^{1/3}$$

$$\frac{b}{a}<1-\frac{1}{9}\frac{\omega^2}{J}\left(\frac{3T}{4\pi}\right)^{5/3}$$

where J is the moment of rotation around the Oz-axis.

Stokes' Theorem (Tisserand, 1849)

Suppose that the density distribution inside T is arbitrary and that the level surface S is defined by

$$U=V+\frac{1}{2}\omega^2(x^2+y^2)=C,$$

with the mass M contained inside S. Then by Gauss' theorem,

$$\int_S\frac{\partial V}{\partial n}d\sigma=\int\Delta Vd\tau=-4\pi\int\rho d\tau=-4\pi M$$

in an interior point, and $\Delta V=0$ in an exterior point. We vary the mass distribution inside by keeping the level surface S fixed, and of course with M fixed. Denote the resulting variation of V by V', that of U by U', and that of C by C'. On S, we have $U'=C'$, and

$$\int_S\frac{\partial V'}{\partial n}d\sigma=-4\pi M.$$

Outside S, we have $\Delta V'=0$. Write $W=V-V'$, then on S we have $W=C-C'$, and

$$\int_S\frac{\partial W}{\partial n}d\sigma=0.$$

Outside S, we have

$$\Delta W=0.$$

Draw a sphere Σ with a very large radius R so that the whole of S is contained within the sphere. In the space T' between Σ and S we have by Green's formula, equation 5,

$$\int_{T'}\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2+\left(\frac{\partial W}{\partial z}\right)^2\right]d\tau=\int_\Sigma W\frac{\partial W}{\partial n}d\sigma+\int_S W\frac{\partial W}{\partial n}d\sigma;$$

$$\int_\Sigma W\frac{\partial W}{\partial n}d\sigma$$

tends toward zero in the same order as $1/R$, as $R \to \infty$. Therefore, we obtain

$$\int_{T'} \left[\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 + \left(\frac{\partial W}{\partial z} \right)^2 \right] d\tau = 0.$$

Hence $W =$ constant in T'. Since W becomes zero at infinity, this constant is zero. Therefore, $W = V - V' = 0$, and $V = V'$. Stokes' theorem states that the potential V is determined uniquely by S, ω, and M. Accordingly, U is determined uniquely by S, ω, and M and does not depend on the mass distribution inside S.

Similarly, the direction of the principal axes of inertia, the moments of inertia, and the products of inertia are determined uniquely by S, ω, and M. In fact,

$$I_1 = \int_T x\rho d\tau = -\frac{1}{4\pi} \int_T x \Delta V_i d\tau.$$

By Green's formula,

$$I_1 = \frac{1}{4\pi} \int_S \left(x \frac{\partial V_i}{\partial n} - U_i \frac{\partial x}{\partial n} \right) d\sigma,$$

since $\Delta x = 0$. But on S we have $V_i = V$, $\partial V_i / \partial n = \partial V / \partial n$. Hence, I_1 is determined uniquely by the uniquely determined V given in the theorem. We get similar results for

$$\int_T y\rho d\tau, \qquad \int_T z\rho d\tau,$$

and accordingly the center of mass is determined uniquely. Similarly, we can determine uniquely

$$\int_T xy\rho d\tau, \qquad \int_T yz\rho d\tau, \qquad \int_T zx\rho d\tau, \qquad \int_T (x^2 - y^2)\rho d\tau, \qquad \int_T (y^2 - z^2)\rho d\tau,$$

and

$$\int_T (z^2 - x^2)\,\rho d\tau.$$

These six quantities determine uniquely the principal axes of inertia and the principal moments of inertia. We can determine the value of V at any interior point when S, ω, M, and the value of V on S are given. This is a boundary value problem called *the generalized Neumann-Dirichlet problem* (Crudeli, 1909, 1935).

We know that in the exterior space $\Delta V = 0$. From equation 2a, we have

$$\int_S \left(V_S \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial V_S}{\partial n} \right) d\sigma = 0.$$

We approach a point $Q(x_0, y_0, z_0)$ on S by remembering that $U = V + (\omega^2/2)(x^2 + y^2)$; then

$$\lim \int_S V_S \frac{\partial \frac{1}{r}}{\partial n} d\sigma - \int_S \frac{1}{r_0} \frac{\partial U}{\partial n} d\sigma + \frac{\omega^2}{2} \int_S \frac{1}{r_0} \frac{\partial (x^2+y^2)}{\partial n} d\sigma = 0.$$

By equation 7a we have

$$\lim \int_S V_S \frac{\partial \frac{1}{r}}{\partial n} d\sigma = -2\pi V_0 + \int_S V_S \frac{\partial \frac{1}{r_0}}{\partial n} d\sigma.$$

From equation 18 and $\partial U/\partial n < 0$, we see that

$$\int_S \frac{1}{r_0} \frac{\partial U}{\partial n} d\sigma = -\frac{4\pi\rho - 2\omega^2}{\rho} V_0.$$

Hence,

$$2(\omega^2 - \pi\rho) V_0 - \rho \int_S V_S \frac{\partial \frac{1}{r_0}}{\partial n} d\sigma = \rho \frac{\omega^2}{2} \int_S \frac{1}{r_0} \frac{\partial (x^2+y^2)}{\partial n} d\sigma.$$

If $\omega^2 \neq \pi\rho$, then

$$V_0 - \frac{\rho}{2(\omega^2 - \pi\rho)} \int_S V_S \frac{\partial (1/r_0)}{\partial n} d\sigma = F,$$

$$F = \frac{\rho\omega}{4(\omega^2 - \pi\rho)} \int_S \frac{1}{r_0} \frac{\partial (x^2+y^2)}{\partial n} d\sigma.$$

Thus this reduces to the integral equation of the Fredholm type

$$\Omega(u, v) + \mu \int_S \Omega(u_S, v_S) \frac{\cos (r, n)}{2\pi r_0^2} d\sigma = F(u, v).$$

If $|\mu| <$, then this equation has no eigenvalue (Plemelj, 1904, 1907, 1911; Kneser, 1922; Kellogg, 1929; Goursat, 1923). Poincaré's inequality states that $\omega^2/\pi\rho < 2$. Hence,

$$\mu = \frac{\pi\rho}{\omega^2 - \pi\rho} = \frac{1}{(\omega^2/\pi\rho) - 1}, \qquad \text{and} \qquad |\mu| < 1.$$

If μ is an eigenvalue, then we denote by $Y_1, \ldots Y_m$ the linearly independent solutions of the homogeneous integral equation

$$Y + \mu \int Y_S \frac{\cos (r, n)}{2\pi r_0^2} d\sigma = 0.$$

It can be proved that

$$\int_S Y_k F d\sigma = 0 \qquad (k = 1, 2 \ldots, m).$$

Hence, even when μ is an eigenvalue, the solution of the nonhomogeneous integral equation exists, provided that there are m solutions instead of one. This is the case of bifurcation.

Thus we can determine V when S, ω, M, and the value of V on S are given. This is called the Stokes problem.

The problem of obtaining the surface S when the values of g over the whole surface S are known can be discussed in a similar manner (Brillouin, 1925; Mineo, 1927, 1933).

This problem of determining S when the distributions of g and dg/dn at any point over the unknown surface S are given, and where the direction of the normal is known by astronomical observations, has an important bearing in geodesy. We do not know yet what is the real figure of the earth; we don't even know where the real center of the earth is. The present method of calculation consists of successive approximations, starting with Bessel's or Hayford's spheroid. This is the problem for geodesic satellites which O'Keefe, Kaula, and others are working to solve. It is the inverse of Neumann-Dirichlet's generalized boundary value problem. Yet this is one of the unsolved problems in mathematical physics (cf., Gunther, 1934). However, recent observations of earth satellites are providing a means for determining the observers' coordinates relative to the mass-center of the earth.

Hamy (1887, 1889) proved that we can determine the ellipticity uniquely when the law of density is given; he also proved that the equilibrium figure of a rotating heterogeneous mass cannot be an ellipsoid with three unequal axes but be an ellipsoid of revolution. Poincaré (1885, 1902) proved that, if the surfaces of separation of such a heterogeneous mass are ellipsoids, then they must be all confocal. Similar problems were investigated by Radau (1885), Callandreau (1889), Véronnet (1912), Dive (1926, 1927, 1930), Wavre (1927, 1928, 1929), and Merlin (1927, 1930). In particular Wavre (1932) led the problem to the solution of an integral equation of Fredholm's type, and Dive (1926, 1927) extended Hamy's theorem.

Limit of Angular Velocity

If we consider a body rotating with angular velocity ω around a fixed axis, then the moment of inertia around that axis is $J=\int \rho d\tau(x^2+y^2)$. When the body is deformed such that the projection of the displacement of a surface element $d\sigma$ on the normal is ζ, then J varies by the amount $dJ=\int \zeta\rho d\sigma(x^2+y^2)$.

If we let $U=V+(\omega^2/2)\,(x^2+y^2)$, then the Newtonian potential energy is

$$W=\int \frac{\rho V}{2}\,d\tau,$$

where

$$V=\int \frac{d\tau}{r}.$$

Then,

$$dW+\frac{\omega^2}{2}\,dJ=\int \zeta\left[V+\frac{\omega^2}{2}\,(x^2+y^2)\right]\rho\delta\sigma=\int U\zeta\rho d\sigma.$$

In equilibrium, $U=\text{constant}=U_0$ on the surface S. Then,

$$dW+\frac{\omega^2}{2}\,dJ=U_0\int \zeta\rho d\sigma=U_0\rho dT.$$

If the body is deformed similarly to itself, then W and J vary as $T^{5/3}$. Hence,

$$\frac{dW+\frac{\omega^2}{2}dJ}{W+\frac{\omega^2}{2}J}=\frac{5}{3}\frac{dT}{T};$$

therefore

$$\frac{3}{5}\rho U_0 T=W+\frac{\omega^2}{2}J.$$

Now $\Delta U=2\omega^2+\Delta V=2\omega^2-4\pi\rho$. Since we have $2\omega^2\leqslant 4\pi\rho$ from Poincaré's inequality, we must have $\Delta U<0$. Thus U cannot have a minimum inside T. Since $U=U_0$ on S, we should have $U>U_0$ inside T. If $\Delta U=0$, then $U=U_0$, and if $\Delta U>0$, then $U<U_0$ in the whole interior.

Consider the integral $\int U\rho d\tau=2W+(\omega^2/2)J$. As $\Delta U \lesseqgtr 0$, the left-hand side

$$\int U\rho d\tau \gtreqless \rho\int U_0 d\tau=\rho U_0\int d\tau=\rho U_0 T=\frac{5}{3}\left(W+\frac{\omega^2}{2}J\right).$$

Hence we obtain the theorem

$$\Delta U<0,\qquad \frac{5}{3}\left(W+\frac{\omega^2}{2}J\right)<2W+\frac{\omega^2}{2}J,\qquad W>\omega^2 J;$$

$$\Delta U=0,\qquad \frac{5}{3}\left(W+\frac{\omega^2}{2}J\right)=2W+\frac{\omega^2}{2}J,\qquad W=\omega^2 J;$$

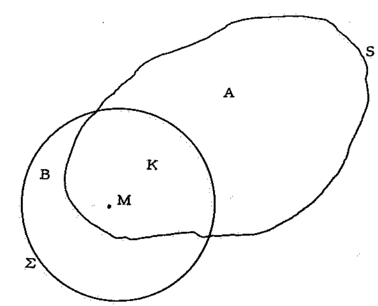
$$\Delta U>0,\qquad \frac{5}{3}\left(W+\frac{\omega^2}{2}J\right)>2W+\frac{\omega^2}{2}J,\qquad W<\omega^2 J.$$

Suppose that ω varies continuously and the figure deforms continuously if ρ remains unchanged. We have seen that

$$dW+\frac{\omega^2}{2}dJ+\omega J d\omega=\frac{3}{5}\rho T dU_0,\qquad T=\text{constant}.$$

Since the figure is in equilibrium, $W+(\omega^2/2)J$ is a maximum or a minimum if $dW+(\omega^2/2)dJ=0$. The remaining part is $\omega J d\omega=(3/5)TdU_0$. Hence $dU_0/d\omega>0$, and U_0 increases as ω increases.

Since V is a Newtonian potential, we have $V=U_0-(\omega^2/2)(x^2+y^2)$ on S. If the surface S intersects the rotation axis, then U_0 is the Newtonian potential at the pole. As ω increases, the potential increases, but U_0 cannot be greater than $2\pi R^2$, where R is the radius of the sphere of the same volume as the body T.



If we draw a sphere Σ of radius R around the point in question, we then have three domains, B, K, and A. The potential of A at $M<A/R$, and the potential of B at $M>B/R=A/R$. But the potential of $S=$ the potential of $K+$ the potential of A, and the potential of $\Sigma=$ the potential of $K+$ the potential of B. Accordingly, the potential of $\Sigma>$ the potential of S. Thus $2\pi R^2>V$. The potential of a sphere is $2\pi R^2\rho$, because

$$\frac{4\pi R^2 dR}{R}\rho.$$

Now dividing $W+(\omega^2/2)J=(3/5)\rho U_0 T$ and $\omega J d\omega=(3/5)\rho T dU_0$ side by side gives,

$$\frac{\omega J d\omega}{W+\dfrac{\omega^2}{2}J}=\frac{dU_0}{U_0}.$$

As ω increases indefinitely, there would be a time when ω^2 finally exceeds π. Then, as $W<\omega^2 J$, we must have

$$\frac{\omega^2 J}{W+\dfrac{\omega^2}{2}J}>\frac{2}{3}.$$

Hence

$$\frac{2}{3}\frac{d\omega}{\omega}<\frac{dU_0}{U_0}.$$

If ω increases indefinitely, then so must U_0. But $U_0 \leqslant 2\pi R^2$. Hence either ω must stop increasing, or the equilibrium figure does not cross the rotation axis. The latter case is of an annular or a detached double-star form. In the former case, there is a limit for the angular velocity ω. For the spheroidal figure of equilibrium we have $\omega<4\pi\times 0.112$, and for the ellipsoidal figure of equilibrium we have $\omega<4\pi\times 0.093$. From this point of view, it is possible that there exists a succession of equilibrium figures passing several maxima and minima of ω in succession. In such cases, the foregoing reasoning cannot be applied, and ω can increase indefinitely. This reasoning holds only in a successive interval between a maximum and the consecutive or the preceding minimum (Poincaré, 1902).

As a corollary, the axis of rotation of an equilibrium figure with a sufficiently large value of ω does not intersect the free surface of the figure. It may be either of annular form or of detached double-star form.

Rotation Axis

We shall now examine whether there may exist equilibrium figures with nonuniform rotation. The center of mass is supposed to be at rest; it makes no difference whether the fluid is solidified or not. If the fluid is solidified, then the motion around the center of mass is a Poinsot rigid-body rotation. By the principle of D'Alembert, the virtual work done by any displacement compatible with the constrained motion is zero. The constraint in this case is the incompressibility of the fluid, and this is expressed by

$$\frac{\partial}{\partial x}\delta x+\frac{\partial}{\partial y}\delta y+\frac{\partial}{\partial z}\delta z=0.$$

There are three kinds of virtual displacement:

(1) The displacement of the whole mass as a solid; this is Poinsot's motion around the center of mass.

(2) Deformation of the body.

(3) Molecules are displaced on the surface of constant density; since the surface of equal density is an equipotential surface, we have

$$\frac{\partial\rho}{\partial x}\,\delta x+\frac{\partial\rho}{\partial y}\,\delta y+\frac{\partial\rho}{\partial z}\,\delta z=0.$$

The surface of constant density remains unchanged by this displacement. Hence the external shape remains unchanged. Let us take a point of mass m with coordinates x, y, z. Let the components of the instantaneous rotation around Ox, Oy, Oz be ω_1, ω_2, ω_3; then

$$\dot{x}=\omega_3 y-\omega_2 z,\qquad \dot{y}=\omega_1 z-\omega_3 x,\qquad \dot{z}=\omega_2 x-\omega_1 y,$$

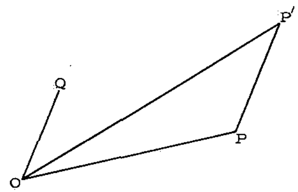
and

$$\ddot{x}=(\omega_3\dot{y}-\omega_2\dot{z})+(\dot{\omega}_3 y-\dot{\omega}_2 z),$$

$$\ddot{y}=(\omega_1\dot{z}-\omega_3\dot{x})+(\dot{\omega}_1 z-\dot{\omega}_3 x),$$

$$\ddot{z}=(\omega_2\dot{x}-\omega_1\dot{y})+(\dot{\omega}_2 x-\dot{\omega}_1 y).$$

As in relative equilibrium, there should be equilibrium between the attraction force and the force of inertia $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$. Suppose that the rotation axis is along OP $(\omega_1, \omega_2, \omega_3)$ at t and along OP' at $t+dt$. Let $OQ=PP'/dt$, and suppose that it takes the limiting position OR as $dt\to 0$. The projection of OR on the three coordinate axes is $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$. If we take OR as the Oz-axis, then $\dot{\omega}_1=\dot{\omega}_2=0$. The inertial force due to angular acceleration has components $\dot{\omega}_3 y, -\dot{\omega}_2 z, 0$. We apply D'Alembert's principle to this virtual displacement. Since the form of the body remains unchanged, the work done by the attraction is zero. The work done by the centrifugal force is $(\omega^2/2)\delta J$, where $\omega^2=\omega_1^2+\omega_2^2+\omega_3^2$, and, since J remains unchanged this work becomes zero. The remaining part $\dot{\omega}_3\Sigma m(y\delta x-x\delta y)$ must be zero in equilibrium. Now we can choose the virtual displacement so that $\Sigma m(y\delta x-x\delta y)$ is not zero. It is sufficient to consider a current around Oz. This sum is equal to the area on the xOy plane bounded by the projection of this current on that plane, and is not zero. Hence $\dot{\omega}_3$ must be zero, that is, the motion must be uniform. Thus it is impossible to have an equilibrium figure of nonuniform rotation (Poincaré, 1902).



Furthermore, the rotation axis must be the axis with the largest principal moment of inertia; that is, the smallest axis of the ellipsoid of inertia. This can be proved from the study of the rotation of a solid body (Pizzetti, 1913).

LIAPOUNOV'S THEOREM

Liapounov's Proof

Liapounov's theorem states that a sphere is the only stable equilibrium figure at rest. This theorem is said to have been discovered by Giessen (1872). Poincaré proved the theorem in his lectures (1902) on the equilibrium figures by referring to an electrostatically charged sphere, on the basis of Dirichlet's criterion of stability. Suppose that the free surface of a body T is an equipotential surface $V=V_0$; then the potential at a point is

$$V=\int_T \frac{\rho\, d\tau}{r},$$

where V is continuous on the free surface and zero at infinity. We have

$$\int_T \Delta V d\tau = -4\pi \int_T \rho d\tau = -4\pi M.$$

Moreover,

$$\int_S \frac{\partial V}{\partial n_e} d\sigma = -4\pi M$$

and $\partial V/\partial n_e < 0$. Now consider the distribution of an electrically charged layer of total charge M in a suitable unit in equilibrium such that the potential V' on the surface is V_0, the potential in the outside space is V, and the potential inside the surface is $V' = V_0$. At infinity, $V' = 0$. The potentials V and V' both satisfy the Laplace equation, and both are equal to V_0 on the free surface. Hence, because of the property of a potential, these two functions should coincide. Poincaré proved the lemmas that the potential of a sphere at its center is the maximum of the potential of any body of equal volume, that a sphere has the minimum electrical capacity among bodies of equal volume, that the electrical capacity is a minimum for a sphere, and that $W = \int (\rho V/2) d\tau$ is a maximum for a sphere among all bodies of equal volume (Poincaré, 1887, 1902).

Liapounov (1884, 1904) proved the theorem with his stability criterion. Denote the moments of inertia by

$$S_x = \int_T (y^2 + z^2) d\tau, \qquad S_y = \int_T (z^2 + x^2) d\tau, \qquad S_z = \int_T (x^2 + y^2) d\tau,$$

$$P_x = \int_T yz d\tau, \qquad P_y = \int_T zx d\tau, \qquad P_z = \int_T xy d\tau,$$

$$D = S_x S_y S_z - S_x P_x^2 - S_y P_y^2 - S_z P^2 - 2P_x P_y P_z,$$

and

$$V = \rho \int_T \frac{d\tau'}{r}, \qquad T_m = \frac{1}{2} \int_T (u^2 + v^2 + w^2) d\tau$$

with the condition that

$$\int_T x d\tau = \int_T y d\tau = \int_T z d\tau = \int_T u d\tau = \int_T v d\tau = \int_T w d\tau = 0,$$

where u, v, w denote the components of the displacement. Denote by $J\rho$ the moment of momentum around the center of inertia, and write

$$\Pi = \frac{1}{2}\left(J^2 \frac{S_x S_y - P_z^2}{D} - \int_T V d\tau\right).$$

According to Liapounov, the necessary and sufficient condition of equilibrium is $\delta\Pi = 0$. If Π is a minimum, then the equilibrium is stable (Lejeune-Dirichlet). This criterion of Liapounov contains the criterion of Poincaré, $(1/2)(J^2 - \int_T V d\tau) =$ minimum, and can define stability, whereas Poincaré's criterion can say nothing about stability.

Denote by n the shortest distance between a point on the surface of an equilibrium figure and its corresponding point on the surface of the distorted figure, counted toward the outer normal. Let $n = n_1 + \delta n$ and let

$$\int_S \delta n d\sigma = 0, \qquad \int_S x\delta n d\sigma = \int_S y\delta n d\sigma = \int_S z\delta n d\sigma = 0, \tag{21}$$

and

$$U = V + \frac{\omega^2}{2}(x^2+y^2).$$

Liapounov obtained

$$\left.\begin{aligned} \delta^2\Pi &= -\int_S \frac{\partial U}{\partial n}(\delta n)^2 d\sigma - \rho \int_S \int \frac{\delta n \delta n' d\sigma d\sigma'}{r} + \Omega + \Psi, \\ \Omega &= \frac{\omega^2}{S_z}\left(\int_S (x^2+y^2)\,\delta n d\sigma\right)^2, \\ \Psi &= \frac{\omega^2}{S_x}\left(\int_S xz\delta n d\sigma\right)^2 + \frac{\omega^2}{S_y}\left(\int_S yz\delta n d\sigma\right)^2. \end{aligned}\right\} \tag{22}$$

If equation 22 is positive for all displacements δn satisfying equation 21, and if $\delta n = -[x \cos(n, y) = y \cos(n, x)]\theta_z$ for $\omega \neq 0$; and if $\delta n = -[y \cos(n, z) - z \cos(n, y)]\theta_x - [z \cos(n, x) - x \cos(n, z)]\theta_y - [x \cos(n, y) - y \cos(n, x)]\theta_z$ for $\omega = 0$, then the equilibrium figure is stable, where θ_x, θ_y, θ_z are independent of x, y, z.

Let the radius of a sphere be R; then

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = -\frac{4}{3}\pi\rho R$$

takes the form

$$\delta^2\Pi = \rho\left[\frac{4}{3}\pi R \int_S (\delta n)^2 d\sigma - \int\int_S \frac{\delta n \delta n' d\sigma d\sigma'}{r}\right].$$

Let $x = R \sin\theta \cos\psi$, $y = R \sin\theta \sin\psi$, $R = R \cos\theta$, and

$$\delta n = \sum_{m=0}^{\infty} Y_m(\theta, \psi).$$

Then equation 21 gives $Y_0(\theta, \psi) = 0$, $Y_1(\theta, \psi) = 0$, and we have

$$\int_S (\delta n)^2 d\sigma = \sum_{m=0}^{\infty} \int_S (Y_m)^2 d\sigma.$$

Furthermore by the relation

$$\int_T \frac{Y_m(\theta', \psi') d\tau'}{r} = \frac{4\pi R}{2m+1} Y_m(\theta, \psi)$$

we obtain

$$\int\int_S \frac{\delta n \delta n' d\sigma d\sigma'}{r} = 4\pi R \sum_{m=2}^{\infty} \frac{1}{2m+1} \int_S (Y_m)^2 d\sigma.$$

Hence

$$\delta^2\Pi=\frac{4\pi}{3}\rho R\sum_{m=2}^{\infty}\frac{2m-2}{2m+1}\int_S (Y_m)^2 d\sigma.$$

This expression cannot be negative and cannot become zero unless all δn are zero simultaneously. Therefore, a sphere is a stable equilibrium figure.

Isoperimetric Problem of a Sphere

Given a closed curve in space, construct the surface of minimum area enclosed by the curve. This is *Plateau's problem,* one of the isoperimetric problems concerning a sphere (Blaschke, 1916, 1921; Bonnesen, 1929; Tonelli, 1923; Minkowski, 1953; Lichtenstein, 1929). A more general problem is to find the configuration of the maximum of a certain volume integral when the surface area of the configuration is fixed. A closed convex surface which is regular analytic and is of positive curvature everywhere is called an *ovaloid,* the only surface of fixed mean curvature. If we deform an ovaloid continuously and isometrically (*längentreu*), then it displaces as a rigid body (*Starrheit* by Weyl and Blaschke).

Theorem: Among all closed surfaces of given volume, a sphere has the minimum surface area.

For proof, we employ the symmetrization method of Steiner. Suppose that the ovaloid K consists of vertical columns parallel to the z-axis. Displace each column parallel to itself so that the center of mass is on the plane $z=0$. Then K becomes an ovaloid K^* of the same volume symmetric with respect to $z=0$. In order to prove that K^* is an ovaloid, it is sufficient to prove that the straight line joining two points P_1^*, Q_1^* of K^* is inside K^*. Let P_2^*, Q_2^* be the mirror images of P_1^*, Q_1^* with respect to $z=0$, and the points of K corresponding to P_1^*, Q_1^*, P_2^*, Q_2^* of K^* be P_1, Q_1, P_2, Q_2. Then, since K is an ovaloid, the convex quadrangle $P_1Q_1P_2Q_2$ is inside K. As P_1^*, Q_1^*, P_2^*, Q_2^* are obtained by symmetrization from P_1, Q_1, P_2, Q_2, the quadrangle $P_1^*Q_1^*P_2^*Q_2^*$ is inside K^*. Hence $P_1^*Q_1^*$ is inside K^*.

Let $f(x, y)$, $(\geqslant 0)$, be the length of the vertical column passing through the point $(x, y, 0)$. The volume is $T=\iint f(xy)\,dxdy$; function $f(xy)$ is the same for K and K^*. Hence the volume is conserved. That the surface K^* is regular and analytic can be seen easily, and the proof that K^* is of smaller surface area is seen in the following manner:

Divide the surface S of the ovaloid into two parts $\bar{S}$ and $\underline{S}$ by the closed curve on the vertical tangent planes and denote them by the Gaussian parameters u and v.

$$\bar{x}(u, v)=\underline{x}(u, v)=x(u, v), \qquad \bar{y}(u, v)=\underline{y}(u, v)=y(u, v).$$

Let

$$z(u, v; t)=\frac{1+t}{2}\bar{z}(u, v)-\frac{1-t}{2}\underline{z}(u, v),$$

$$\Phi(t)=\iint\sqrt{A^2+B^2+C^2}\,dudv,$$

$$A=\frac{\partial(y, z)}{\partial(u, v)}=\frac{1+t}{2}\bar{A}-\frac{1-t}{2}\underline{A},$$

$$B=\frac{\partial(z, x)}{\partial(u, v)}=\frac{1+t}{2}\bar{B}-\frac{1-t}{2}\underline{B},$$

$$C=\frac{\partial(x, y)}{\partial(u, v)};$$

Φ denotes the area of surface S. It is sufficient to prove that $\Phi(+1)-2\Phi(0)+\Phi(-1)\geqslant 0$. We see that

$$\Phi''(t)=\iint\frac{[A(\bar{B}+\underline{B})-B(\bar{A}+\underline{A})]^2+[(\bar{A}+\underline{A})^2+(\bar{B}+\underline{B})^2]C^2}{4(A^2+B^2+C^2)^{3/2}}\,dudv,$$

and for $-1<t<+1$ we have $\Phi''(t)\geqslant 0$. Hence the area $\Phi(t)$ is decreased by the symetrization. Consequently a sphere has the minimum surface area among all ovaloids of given volume.

The rigorous existence-proof of such a limiting ovaloid, a sphere, with

$$F\geqslant F_n, \qquad \lim_{n\to\infty} F_n=F_\Sigma,$$

where Σ denotes the sphere, was completed by Gross in 1917. Blaschke proved the existence of the limiting figure by referring to the Bolzano-Weierstrass theorem. We can choose from an infinite sequence of uniformly bounded convex bodies a subsequence of convex bodies K_1, K_2, . . . converging to a convex body L, such that

$$L=\lim_{n\to\infty} K_n.$$

Here "uniformly bounded" means that the body is contained in a hexahedron or a sphere.

Blaschke proved the following therorems (1918). Let B be an ovaloid, F its surface area, and E the mean of the distances of any two points P, Q in the domain. Then

$$E=\frac{1}{F^2}\int\int_{BB'}(PQ)dF_PdF_Q.$$

A circle has a minimum E for a given F. Between the integral invariants E and F of B, we have

$$3^4\cdot 5^2\cdot \pi^3E^2-2^{14}\cdot F\geqslant 0.$$

$E\geqslant E^*$, where E^* is a figure obtained from E by symmetrization. The solution, if it exists, should be a cricle.

Let

$$J=\int_B\int_B f(r_{PQ})dF_PdF_Q, \qquad f'(r)<0;$$

J has a maximum value for a circle. For $f(r)=1/r$, we have Liapounov's theorem.

Carleman's Theorem (Carleman, 1919)

(1) If $f(r)>0$ is decreasing, then J has a maximum value for a circle.

(2) If $f(r)>0$ is decreasing and if $\Phi(P)=\int_B f(r_{PQ})dF_Q$ has a constant value on the boundary of domain B, or, if the area of domain B is kept constant, then domain B is a circle for $\delta J=0$.

Lichtenstein (1919) extended the theorem to three dimensions, and also to nonhomogeneous bodies. Consider a body B_0 consisting of a finite number of pieces bounded by analytic surfaces. Any mass may have a point or line in common with another mass. The mass distribution is supposed to be such that the density ρ varies continuously between its minimum ρ' and its maximum ρ''. Let B_ρ be the total mass in B_0 such that its density $\geqslant \rho$. Mass B_ρ may consist of a finite number of pieces, but is bounded by analytic regular surfaces. For $\rho = \rho''$, the total mass B_ρ reduces to a finite number of analytic surfaces, or a line, or a point. Denote the volume of B_ρ by $V(\rho)$; then $V(\rho_1) > V(\rho_2)$ for $\rho_1 < \rho_2$. Furthermore $V(\rho'') = 0$; also, $V(\rho)$ decreases monotonously and continuously with ρ, but may be stepwise discontinuous. Lichtenstein proved that the energy is a minimum for a distribution of B_ρ in the form of concentric spheres among the distributions with a given value of $V(\rho)$ of mass B_0.

A more rigorous mathematical proof was given by Gross (1917) after the method of solving the isoperimetric problem of a sphere by Tonelli (1915) and by Rosenblatt (1920) using Fubini's theorem on the integral of the measure of a set of points.

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CHAPTER II

Ellipsoidal Figures of Equilibrium

POTENTIAL OF AN ELLIPSOID

A point on the surface of an ellipsoid is represented by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0. \tag{23}$$

For a point P outside the space bounded by this ellipsoid we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1>0.$$

Consider

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}-1=0.$$

The left-hand side is positive for $u=0$, decreases continuously as u increases, and becomes -1 for $u=\infty$. Hence it has one and only one positive root. Denote the root by u; the outside potential of the ellipsoid at a point $u(x, y, z)$ determined by the root u with given x, y, z, is

$$V_e=\pi abc\rho\int_u^\infty\left(1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}-\frac{z^2}{c^2+\lambda}\right)\frac{d\lambda}{\sqrt{\varphi(\lambda)}}, \tag{24}$$

where

$$\varphi(\lambda)=(a^2+\lambda)(b^2+\lambda)(c^2+\lambda).$$

The inside potential is

$$V_i=\pi abc\rho\int_0^\infty\left(1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}-\frac{z^2}{c^2+\lambda}\right)\frac{d\lambda}{\sqrt{\varphi(\lambda)}}. \tag{25}$$

The root u is zero on the surface, and the two functions V_e and V_i coincide on the surface. V_e and V_i have both properties of potentials because V_e becomes zero of order $1/R$ at infinity, and its first derivative is continuous and satisfies $\Delta V = -4\pi\rho$ for the whole space, where ρ is supposed to be zero outside the ellipsoid.

Let

$$K=\int_0^\infty \frac{d\lambda}{\sqrt{\varphi(\lambda)}}, \qquad A=\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{\varphi(\lambda)}}, \qquad B=\int_0^\infty \frac{d\lambda}{(b^2+\lambda)\sqrt{\varphi(\lambda)}},$$
$$C=\int_0^\infty \frac{d\lambda}{(c^2+\lambda)\sqrt{\varphi(\lambda)}}.$$

Then

$$V_i=\pi abc\rho(K-Ax^2-By^2-Cz^2); \tag{25a}$$

potential V_i is a maximum at the center, and its value is $\pi abc\rho K$. The equipotential surface is $K-Ax^2-By^2-Cz^2=h$, or $Ax^2+By^2+Cz^2=K-h$ and is homothetic and concentric as h varies. It can be shown that the potential is a maximum at the end of the minor axis and a minimum at the end of the major axis.

The energy of the total mass is

$$W=\frac{1}{2}\rho\int Vd\tau=\frac{1}{2}\pi\rho^2abc\left(K\int d\tau-A\int x^2d\tau-B\int y^2d\tau-C\int z^2d\tau\right)$$
$$=\frac{8}{15}\pi^2\rho^2a^2b^2c^2\int_0^\infty \frac{d\lambda}{\sqrt{\varphi(\lambda)}}. \tag{26}$$

In the case of an oblate (planetary) spheriod, we obtain

$$\left.\begin{aligned}
a&=b=\frac{(\zeta^2+1)^{1/2}}{\zeta}c,\\
A&=B=\frac{1}{abc}[(\zeta^2+1)\zeta\cot^{-1}\zeta-\zeta^2],\\
C&=\frac{1}{abc}2(\zeta^2+1)(1-\zeta\cot^{-1}\zeta),\\
W&=\frac{16}{15}\pi^2\rho^2R^5\left(\frac{\zeta^2+1}{\zeta^2}\right)^{1/3}\zeta\cot^{-1}\zeta, \qquad R=(abc)^{1/3}.
\end{aligned}\right\} \tag{27}$$

If we denote the eccentricity of the meridian section by e, then

$$\left.\begin{aligned}
e^2&=1-\frac{c^2}{a^2}=\frac{1}{\zeta^2+1},\\
abcA&=abcB=\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e-\frac{1-e^2}{e^2},\\
abcC&=\frac{2}{e^2}\left(1-\sqrt{1-e^2}\,\frac{\sin^{-1}e}{e}\right),\\
W&=\frac{16}{15}\pi^2\rho^2R^5(1-e^2)^{1/6}\frac{\sin^{-1}e}{e}
\end{aligned}\right\} \tag{27a}$$

(Somigliana, Morea, Heine, Tedone).

Theorem: A polynomial of the second degree

$$\Phi = -\alpha x^2 - \beta y^2 - \gamma z^2 + \delta,$$

which is positive in a region T, represents a Newtonian potential for the inside of one and only one homogeneous ellipsoid in that region (Dive, 1931a, 1931b).

Ivory's lemma: Take two homofocal ellipsoids E and E_1:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad \frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}+\frac{z^2}{c_1^2}=1,$$

with

$$a_1^2-a^2=b_1^2-b^2=c_1^2-c^2.$$

Consider a point $A(x, y, z)$ of E and a point $A_1(x, y, z)$ of E_1, such that

$$x_1=\frac{a_1}{a}x, \qquad y_1=\frac{b_1}{b}y, \qquad z_1=\frac{c_1}{c}z,$$

and another pair of such points $B(x', y', z')$ and $B_1(x_1', y_1', z_1')$; then we have $AB_1=A_1B$. Such a pair of points is said to be *corresponding.*

Ivory's theorem: Let P and P_1 be two corresponding points on two homogeneous confocal ellipsoids $E(abc)$ and $E_1(a_1b_1c_1)$. The x-component of the attraction due to E at P and the x-component of the attraction due to E_1 at P_1 are in the ratio $(bc)/(b_1c_1)$.

Chasle's corollary: Take two confocal homogeneous ellipsoids of equal mass. The potential at a point of the first ellipsoid due to the second ellipsoid is equal to the potential at the corresponding point of the second ellipsoid due to the first ellipsoid.

Maclaurin's corollary: The attraction at an external point of a homogeneous ellipsoid is in the same direction as the attraction due to the homofocal ellipsoid passing through that point, and their strengths are in the ratio of the masses of the two ellipsoids.

Newton's corollary: The homogeneous shell contained between two homothetic concentric ellipsoids does not affect any attractive force at an inside point in the cavity, or, in other words, the volume potential of a homogeneous ellipsoidal shell is constant inside the cavity.

The converse is also true. In order that the potential in the cavity be constant, the two homothetic surfaces should be ellipsoids (Dive, 1931b, 1932a, 1932b).

Duhamel's corollary: A spherical shell contained between two concentric homogeneous spheres does not affect any attractive force at any point inside the cavity. This is the basis of the famous experiment of Cavendish on electricity. Cavendish took an electrometer into a highly charged, spherical cavity but could find no change of electric force inside.

Conversely, Robin's problem is to find the distribution of electricity on the given surface S for which the interior potential is constant (Gunther, 1934).

Poisson's theorem: The attraction at an external point due to a homogeneous homothetic ellipsoidal shell is directed toward the internal axis of the cone with the point as the vertex and in contact externally with the homogeneous ellipsoid (Gray, 1913).

The proof of these classical theorems are in the books by Tisserand (1891), Thomson and Tait (1883), Pizzetti (1913), or Poincaré (1899).

Stokes' theorem: Given the surface S of a mass in equilibrium and its mass M and its angular velocity ω, we can determine the external potential V. In fact

$$\Delta V_e = 0, \qquad V_i + \frac{\omega^2}{2}(x^2+y^2) = \text{constant}$$

on the given surface S, and $\lim_{r=\infty} rV = M$. The function $U = V + (\omega^2/2)(x^2+y^2)$ is determined by Dirichlet's problem.

Corollary: We can also determine the principal moments of inertia together with the directions. From this theorem, it follows that the mass distribution inside the earth cannot be determined by the gravity measurement on the earth's surface alone or by the perturbation on other planets alone. Even when the rotation velocity ω depends only on the distance from the rotation axis, we can determine the external potential V merely from S, M, and ω (Dive, 1928; Wavre, 1927, 1932b).

Clairaut's theorem: The difference between the relative decrement of gravity from pole to equator and the ellipticity is 5/2 times the ratio of the centrifugal force and gravity at the equator; that is,

$$\frac{g_p - g_e}{g_e} - \frac{a-b}{a} = \frac{5}{2}\frac{\omega^2 a}{g_e}.$$

MACLAURIN SPHEROID (MACLAURIN, 1742; LAPLACE, 1776; LEGENDRE, 1789)

Put $a=b$, $a>b>c$; then

$$V_i = \pi a^2 c\rho \int_0^\infty \left(1 - \frac{x^2+y^2}{a^2+\lambda} - \frac{z^2}{c^2+\lambda}\right) \frac{d\lambda}{(a^2+\lambda)\sqrt{c^2+\lambda}}.$$

Put

$$X = \frac{\partial V}{\partial x} = -Px, \qquad Y = \frac{\partial V}{\partial y} = -Py, \qquad Z = \frac{\partial V}{\partial z} = -Rz;$$

then

$$P = 2\pi a^2 c\rho \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^2\sqrt{c^2+\lambda}}, \qquad R = 2\pi a^2 c\rho \int_0^\infty \frac{d\lambda}{(c^2+\lambda)^{3/2}(a^2+\lambda)};$$

$$dV = Xdx + Ydy + Zdz = -\frac{d}{2}[P(x^2+y^2) + Rz^2], \tag{28}$$

and the condition of relative equilibrium is

$$\frac{p}{\rho} = V + \frac{\omega^2}{2}(x^2+y^2) + \text{constant} = -\frac{1}{2}[(P-\omega^2)(x^2+y^2) + Rz^2] + \text{constant}.$$

The surface of equal pressure is given by

$$(P-\omega^2)(x^2+y^2) + Rz^2 = \text{constant}, \tag{29}$$

which is a rotation figure around Oz. We determine C so that the free surface coincides with

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1,$$

and we have

$$a^2(P-\omega^2)=c^2R=C. \tag{30}$$

This gives the relation beyween C and ω.

At first we see that $c<a$. In fact, $a^2P-c^2R=a^2\omega^2$, or, substituting the foregoing expressions, we see that the left-hand member becomes

$$(a^2-c^2)\int_0^\infty \frac{\lambda d\lambda}{(a^2+\lambda)(c^2+\lambda)\sqrt{\varphi(\lambda)}}>0.$$

Next we write $a^2e^2=a^2-c^2$, or $ae/c=f$. We obtain from equation 30

$$(1+f^2)P-R=\omega^2(1+f^2),$$

wherc P and R depend on the ratio of the two axes. Put $\lambda=c^2t$, then

$$P=2\pi\rho(1+f^2)\int_0^\infty \frac{dt}{(1+f^2+t)\sqrt{1+t}},$$

or, after computation,

$$\left.\begin{aligned} P&=2\pi\rho\,\frac{1+f^2}{f^3}\left(\tan^{-1}f-\frac{f}{1+f^2}\right),\\ R&=2\pi\rho\,\frac{1+f^2}{f^3}\,(2f-2\tan^{-1}f); \end{aligned}\right\} \tag{31}$$

hence

$$\frac{\omega^2}{2\pi\rho}=\frac{(3+f^2)\tan^{-1}f-3f}{f^3}=h.$$

Furthermore

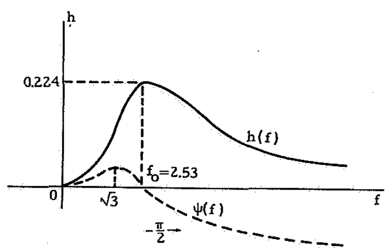
$$M=\frac{4}{3}\,\pi a^2c\rho=\frac{4}{3}\,\pi\rho c^3(1+f^2)=\frac{4}{3}\,\pi\rho\,\frac{a^3}{\sqrt{1+f^2}}. \tag{32}$$

the function h has a maximum at $f=f_0=2.53\ldots$; and

$$\frac{dh}{df}=\frac{9+f^2}{f^4}\,\psi(f),$$

$$\psi(f)=\frac{7f^2+9}{(1+f^2)(9+f^2)}f-\tan^{-1}f,$$

$$\psi'(f)=\frac{8f^4(3-f^2)}{(1+f^2)^2(9+f^2)^2}.$$



If $\omega^2/(2\pi\rho)<0.224$, we obtain two Maclaurin spheroids; if $\omega^2/(2\pi\rho)=0.224$, we obtain only one Maclaurin spheroid, and if $\omega^2/(2\pi\rho)>0.224$, there is no Maclaurin spheroid. There is no natural celestial body for $f > 2.53$. When $\omega \to 0$ or $h \to 0$, then one of the roots tends to zero, that is, the figure tends to a sphere. If f is large, then e is small and a is large, and the figure tends to a flat circular disc of large radius:

$$c^3 = \frac{3M}{4\pi\rho}\frac{1}{1+f^2}, \qquad a^3 = \frac{3M}{4\pi\rho}\sqrt{1+f^2}.$$

JACOBI ELLIPSOID (JACOBI, 1834; LIOUVILLE, 1834; SMITH, 1838; PLANA, 1853; DARWIN, 1887)

Let

$$X = \frac{\partial V}{\partial x} = -Px, \qquad Y = \frac{\partial V}{\partial y} = -Qy, \qquad Z = \frac{\partial V}{\partial z} = -Rz,$$

where

$$P = 2\pi abc\rho \int_0^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{\varphi(\lambda)}}, \text{ etc.}; \tag{33}$$

then the condition for equilibrium is

$$\frac{p}{\rho} = -\frac{1}{2}[(P-\omega^2)x^2 + (Q-\omega^2)y^2 + Rz^2] + \text{constant}. \tag{34}$$

The surface

$$(P-\omega^2)x^2 + (Q-\omega^2)y^2 + Rz^2 = \text{constant} \tag{35}$$

is homothetic and concentric of the second degree. In order that this surface should coincide with the free surface of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, we should have

$$a^2(P-\omega^2) = b^2(Q-\omega^2) = c^2R = C, \tag{36}$$

whence we obtain the conditions

$$\omega^2 = \frac{a^2P - b^2Q}{a^2+b^2} \tag{37}$$

and

$$a^2b^2(P-Q) + c^2(a^2-b^2)R = 0. \tag{38}$$

From equation 37 we obtain

$$\frac{\omega^2}{2\pi\rho} = abc\int_0^\infty \frac{\lambda\, d\lambda}{(a^2+\lambda)(b^2+\lambda)\sqrt{\varphi(\lambda)}}.$$

Put

$$\lambda = c^2\kappa, \qquad \frac{c^2}{a^2} = s, \qquad \frac{c^2}{b^2} = t;$$

then the conditions are

$$\frac{\omega^2}{2\pi\rho} = st\int_0^\infty \frac{\kappa\, d\kappa}{(1+s\kappa)(1+t\kappa)\Delta}, \qquad \Delta = \sqrt{(1+s\kappa)(1+t\kappa)(1+\kappa)}, \tag{39}$$

and

$$(1-s-t)\int_0^\infty \frac{\kappa\, d\kappa}{\Delta^3} - st\int_0^\infty \frac{\kappa^2\, d\kappa}{\Delta^3} = 0. \tag{40}$$

Also we obtain

$$M=\frac{4}{3}\pi abc\rho=\frac{4}{3}\pi\rho\frac{c^3}{\sqrt{st}}.$$

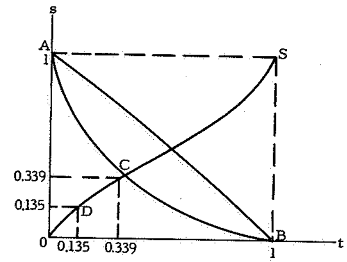
It should be noticed that $s+t<1$, and hence $s<1$, $t<1$, and accordingly $c^2<a^2$, $c^2<b^2$, and c is the shortest axis.

Now we determine s and t with a given value of ω, or conversely determine ω with given values of s and t (Tisserand, 1891).

Let $h=\omega^2/2\pi\rho$ as before; then

$$h=\varphi(s,\,t)\equiv st\int_0^\infty\frac{\kappa d\kappa}{(1+s\kappa)(1+t\kappa)\Delta},$$

$$0=\psi(s,\,t)\equiv(1-s-t)\int_0^\infty\frac{\kappa d\kappa}{\Delta^3}-st\int_0^\infty\frac{\kappa^2d\kappa}{\Delta^3}.$$



It can be shown (Appell, 1921b) that

(1) $s,t>0$;

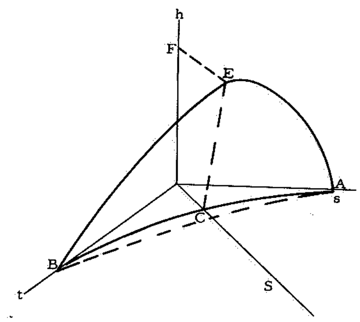
(2) $\gamma(s,\,t)$ is symmetric with respect to s and t;

(3) $s=0$ at $t=1$;

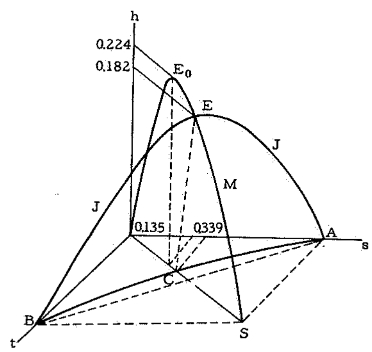
(4) t decreases with increasing s;

(5) for a given value of $0<s<1$, there is only one value of $0<t<1$;

(6) C is at $s=t=t_0=0.3396$, and the value of h at E is $h_0=0.1871$.



The condition for the existence of a Jacobi ellipsoid is $\omega^2(2\pi\rho)<0.1871$. We obtain two Jacobi ellipsoids for $h<h_0=0.187$, but the two are identical. For $h=0.187$ we obtain only one ellipsoid with $a=b$. The Jacobi ellipsoid turns into a Maclaurin spheroid at E. This is the bifurcation point of the two series of equilibrium figures. There is no Jacobi ellipsoid for $h>0.187$. For $\omega\rightarrow 0$, the major axis increases indefinitely, and the middle and the minor axes tend to zero but their ratio to 1; that is, the figure is infinitely long and needle-like with a circular section. Thus we obtain three figures—a sphere, an infinitely thin circular disc, and an infinitely long needle. The theory of linear series has been discussed by Poincaré (1885a, 1885b) with the idea of the exchange of stability at the bifurcation point. A Maclaurin spheroid is stable in the linear series S to E but is no longer stable in the linear series E to O. On the other hand a Jacobi ellipsoid in the linear series E to A and E to B is stable. The stability character is exchanged at point E.



COSMOGONICAL PROBLEM

From a cosmogonical point of view, we should take $\mu=J\omega$ as the parameter. A nebula with a vanishingly small ω may develop ω by shrinking (Laplace, Liouville). For a Maclaurin spheroid,

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$$J=\frac{2}{5}Ma^2, \qquad h=\frac{\omega^2}{2\pi\rho}=\frac{\mu^2}{2\pi\rho J^2}=\frac{25\mu^2}{4M^2a^4}\frac{1}{2\pi\rho},$$

$$\frac{50\mu^2}{3M^3}\left(\frac{4\pi\rho}{3M}\right)^{1/3}=4(1+f^2)^{2/3}\left[\frac{(3+f^2)\tan^{-1}f-3f}{f^3}\right].$$

Put $k=\varphi(f)\equiv 4h(1+f^2)^{2/3}$, which is supposedly given. For a Jacobi ellipsoid,

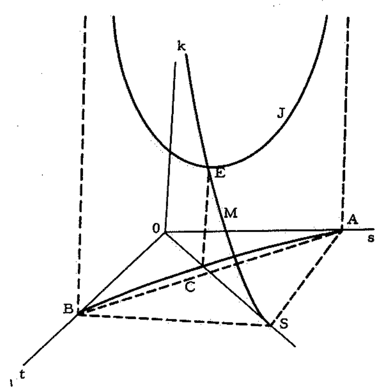
$$J=\frac{M}{5}(a^2+b^2)=\frac{Mc^2}{5}\left(\frac{1}{s}+\frac{1}{t}\right),$$

$$\omega^2=\frac{\mu^2}{J^2}=\frac{25\mu^2}{M^2c^4}\left(\frac{st}{s+t}\right)^2,$$

$$\frac{\omega^2}{2\pi\rho}=\frac{50\mu^2}{3M^3}\left(\frac{4\pi\rho}{3M}\right)^{1/3}\frac{(st)^{4/3}}{(s+t)^2}\equiv\varphi(s,t)=h,$$

$$k=\frac{(s+t)^2}{(st)^{4/3}}\varphi(s,t),$$

$$\psi(s,t)=0;$$

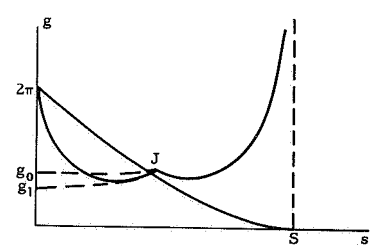


k is a function of ρ and μ. If $\rho=$ constant, then k shows the variation of μ; if $\mu=$ constant, k shows the variation of ρ. The conclusion is that there exist one or two ellipsoidal figures of equilibrium for any given value of μ. Véronnet (1919a, 1919b) varied a instead of ρ in his work.

For a Maclaurin spheroid, we have

$$\left(\frac{4\pi\rho}{3M}\right)^{1/3}=\frac{(1+f^2)^{1/6}}{a}; \qquad g\equiv\frac{50}{3M^3}\frac{\mu^2}{a}=4h\sqrt{1+f^2}=4\sqrt{1+f^2}\left[\frac{(3+f^2)\tan^{-1}f-3f}{f^3}\right].$$

For a Jacobi ellipsoid

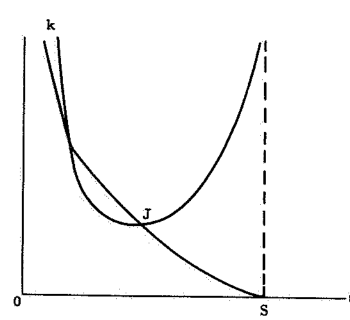


$$g=\frac{50}{3M^3}\frac{\mu^2}{a}=k\frac{s^{1/3}}{t^{1/6}}=\frac{(s+t)^2}{st}\frac{h}{\sqrt{t}}.$$

He obtained the figures

$$g_0=1.28, \qquad t_0=s_0=0.340;$$

$$g_1=1.18, \qquad s_1=0.24, \quad t_1=0.45.$$



For cosmogonical applications see Appell (1922); Véronnet (1926a, 1926b, 1933); Jeffreys (1952); Poincaré (1913); Chamberlain and Moulton (1909); Narliker (1934); Nölke (1930, 1932); Filon (1932). These discussions are on statical grounds, and should be dynamical (Jeans, 1917, 1919).

In the last few years several books on heterogeneous equilibrium figures have appeared, intended for application to planetary and double-star problems, e.g., Wavre (1932a), Kopal (1960), Jardetzky (1958), and Meffroy (1962). The most interesting feature is the difference of the surface of equal pressure—isobaric surface—and the surface of equal density—isoteric surface—, the idea initiated by Bjerknes in meterology and worked out by Wavre (1932a) in stellar applications. Recently Chandrasekhar (1962) based his discussion on his theory of super-potentials and on the virial theorem.

HOMOGRAPHIC MOTION

Consider a nonrigid body motion of a homogeneous, fluid mass and keep its ellipsoidal figure of equilibrium unchanged; that is, allow internal motion of fluid mass by fixing the external shape of the figure. Such a motion is called *homographic.* In other words, the external shape is an ellipsoid rotating around its own axis, but the motion of the fluid is rotational with a different angular velocity to that of the external shape. At first, solidify the fluid in the form of an ellipsoid and fit the fluid inside a vessel of the same form and size, and let it rotate. Then melt the fluid and let the vessel have an additional angular velocity. Finally, remove the vessel (Pizzetti, 1913; Basset, 1888; Greenhill; 1882; Lamb, 1959).

A molecule of the fluid which was at a point (ξ, η, ζ) at $t=0$ referred to the fixed axes in space is supposed at $t=t$, to be at

$$\left|\begin{array}{l} x=\ell\xi+m\eta+n\zeta, \\ y=\ell'\xi+m'\eta+n'\zeta, \\ z=\ell''\xi+m''\eta+n''\zeta, \end{array}\right\} \qquad D=\begin{vmatrix} \ell & m & n \\ \ell' & m' & n' \\ \ell'' & m'' & n'' \end{vmatrix}=1,$$

where the direction cosines $\ell, m, n, \ell' m' n', \ell'' m'' n''$ are functions of t.

The Lagrangian equations of motion are

$$\xi\left(\ell\frac{d^2\ell}{dt^2}\right)+\eta\left(\ell\frac{d^2m}{dt^2}\right)+\zeta\left(\ell\frac{d^2n}{dt^2}\right)=\frac{\partial V}{\partial\xi}-\frac{1}{\rho}\frac{\partial p}{\partial\xi},$$

$$\xi\left(m\frac{d^2\ell}{dt^2}\right)+\eta\left(m\frac{d^2m}{dt^2}\right)+\zeta\left(m\frac{d^2n}{dt^2}\right)=\frac{\partial V}{\partial\eta}-\frac{1}{\rho}\frac{\partial p}{\partial\eta},$$

$$\xi\left(n\frac{d^2\ell}{dt^2}\right)+\eta\left(n\frac{d^2m}{dt^2}\right)+\zeta\left(n\frac{d^2n}{dt^2}\right)=\frac{\partial V}{\partial\zeta}-\frac{1}{\rho}\frac{\partial p}{\partial\zeta},$$

where

$$\left(\ell\frac{d^2\ell}{dt^2}\right)=\ell\frac{d^2\ell}{dt^2}+\ell'\frac{d^2\ell'}{dt^2}+\ell''\frac{d^2\ell''}{dt^2}, \text{ etc.}$$

Differentiating these equations with respect to ξ, η, ζ gives

$$\left(\ell\frac{d^2l}{dt^2}\right)=\frac{\partial^2 V}{\partial\xi^2}-\frac{1}{\rho}\frac{\partial^2 p}{\partial\xi^2}, \text{ etc.,}$$

and

$$\left(m\frac{d^2n}{dt^2}\right)=\left(n\frac{d^2m}{dt^2}\right)=\frac{\partial^2 V}{\partial\eta\partial\zeta}-\frac{1}{\rho}\frac{d^2p}{\partial\eta\partial\zeta}, \text{ etc.}$$

We obtain three integrals from the latter equations:

$$m\frac{dn}{dt}-n\frac{dm}{dt}+m'\frac{dn'}{dt}-n'\frac{dm'}{dt}+m''\frac{dn''}{dt}-n''\frac{dm''}{dt}=\text{constant.}$$

Assuming that the axis of the principal moment of inertia coincides with the coordinate axis at $t=0$, we obtain three integrals of area

$$\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\rho d\tau=X\left(\ell\frac{d\ell'}{dt}-\ell'\frac{d\ell}{dt}\right)+Y\left(m\frac{dm'}{dt}-m'\frac{dm}{dt}\right)$$
$$+Z\left(n\frac{dn'}{dt}-n'\frac{dn}{dt}\right)=\text{constant, etc.,}$$

and one integral of energy:

$$X\left[\left(\frac{d\ell}{dt}\right)^2\right]+Y\left[\left(\frac{dm}{dt}\right)^2\right]+Z\left[\left(\frac{dn}{dt}\right)^2\right]-\rho\int Vd\tau=\text{constant.}$$

Let the equilibrium figure be

$$\frac{\xi^2}{a_0^2}+\frac{\eta^2}{b_0^2}+\frac{\zeta^2}{c_0^2}=1$$

and

$$\xi=\lambda x+\lambda' y+\lambda'' z,$$
$$\eta=\mu x+\mu' y+\mu'' z,$$
$$\zeta=\nu x+\nu' y+\nu'' z.$$

The free surface at t is also an ellipsoid

$$\frac{(\lambda x+\lambda' y+\lambda'' z)^2}{a_0^2}+\frac{(\mu x+\mu' y+\mu'' z)^2}{b_0^2}+\frac{(\nu x+\nu' y+\nu'' z)^2}{c_0^2}=1,$$

and the potential is

$$V=K-L\xi^2-M\eta^2-N\zeta^2-2L'\eta\zeta-2M'\xi\zeta-2N'\xi\eta,$$

where the coefficients are functions of t. Substituting this in the Lagrangian equations, we see that $\partial p/\partial\xi$, $\partial p/\partial\eta$, $\partial p/\partial\zeta$ are linear homogeneous functions of ξ, η, ζ, and p is of a similar form to V as a function of ξ, η, ζ, and t. But p should be a constant on the ellipsoidal surface; hence

$$p=\rho\sigma\left(1-\frac{\xi^2}{a_0^2}-\frac{\eta^2}{b_0^2}-\frac{\zeta^2}{c_0^2}\right)+\bar{\omega},$$

where σ, $\bar{\omega}$ are functions of t only. Inserting this in the equations for

$$\left(\ell\frac{d^2\ell}{dt^2}\right),\ \left(m\frac{d^2n}{dt^2}\right),\ \text{etc.,}$$

and after some computation, we obtain nine equations of the form

$$\frac{d^2\ell}{dt^2}=\frac{\partial}{\partial\xi}\left(\frac{\partial V}{\partial x}\right)+2\sigma\frac{\lambda}{a_0^2},\quad \frac{d^2m}{dt^2}=\frac{\partial}{\partial\eta}\left(\frac{\partial V}{\partial x}\right)+2\sigma\frac{\lambda''}{b_0^2},\quad \ldots,$$

$$\frac{d^2\ell'}{dt^2}=\frac{\partial}{\partial\xi}\left(\frac{\partial V}{\partial y}\right)+2\sigma\frac{\lambda'}{a_0^2},\quad \ldots,\quad \ldots,$$

$$\ldots,\quad \ldots,\quad \frac{d^2n''}{dt^2}=\frac{\partial}{\partial\zeta}\left(\frac{\partial V}{\partial z}\right)+2\sigma\frac{\nu''}{c_0^2}.$$

With $\Delta V=-4\pi\rho$, we obtain σ from

$$2\sigma\left[\frac{(\lambda^2)}{a_0^2}+\frac{(\mu^2)}{b_0^2}+\frac{(\nu^2)}{c_0^2}\right]=4\pi\rho-\left(\frac{\partial\lambda}{\partial t}\frac{\partial\ell}{\partial t}\right)-\left(\frac{\partial\mu}{\partial t}\frac{\partial m}{\partial t}\right)-\left(\frac{\partial\nu}{\partial t}\frac{\partial n}{\partial t}\right).$$

This equation expresses the surface pressure as a function of the position at t. For a uniform rotation, this equation reduces to

$$2\sigma\left(\frac{1}{a_0^2}+\frac{1}{b_0^2}+\frac{1}{c_0^2}\right)=4\pi\rho-2\omega^2,$$

and

$$\sigma=\frac{gS}{2T\left(\frac{1}{a_0^2}+\frac{1}{b_0^2}+\frac{1}{c_0^2}\right)},$$

where g is the mean surface gravity.

RIEMANN'S PROBLEM (1861)

Suppose that the external axes of the ellipsoid are invariable, but the direction of the axes is variable. Denote the coordinates that coincide with the axes a, b, c of the ellipsoid by x', y', z'. Then with reference to the coordinate axes fixed in space,

$$\left.\begin{aligned} x&=\alpha_1x'+\alpha_2y'+\alpha_3z',\\ y&=\beta_1x'+\beta_2y'+\beta_3z',\\ z&=\gamma_1x'+\gamma_2y'+\gamma_3z'\end{aligned}\right\}.$$

On the other hand,

$$\left.\begin{aligned} x'&=L\xi+M\eta+N\zeta,\\ y'&=L'\xi+M'\eta+N'\zeta,\\ z'&=L''\xi+M''\eta+N''\zeta\end{aligned}\right\}.$$

Thus the general motion of molecules is decomposed into rigid and homographic motion. The latter is an internal motion. The external shape is represented by

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1.$$

From

$$\frac{d^2x}{dt^2}=\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ etc.,}$$

we derive

$$\alpha_1\frac{d^2x}{dt^2}+\beta_1\frac{d^2y}{dt^2}+\gamma_1\frac{d^2z}{dt^2}=\frac{\partial V}{\partial x'}-\frac{1}{\rho}\frac{\partial p}{\partial x'}, \text{ etc.}$$

The left-hand side is the projection of the acceleration of a molecule on the x'-axis. This is composed of three components:

(1) The component of the relative acceleration d^2x'/dt^2, or

$$\xi\frac{d^2L}{dt^2}+\eta\frac{d^2M}{dt^2}+\zeta\frac{d^2N}{dt^2};$$

(2) The component of the acceleration due to deformation

$$x'A_{11}+y'A_{12}+z'A_{13}, \qquad \text{where} \qquad A_{rs}=\alpha_r\frac{d^2\alpha_s}{dt^2}+\beta_r\frac{d^2\beta_s}{dt^2}+\gamma_r\frac{d^2\gamma_s}{dt^2};$$

and

(3) The composite centrifugal acceleration

$$2\left(Q\frac{dz'}{dt}-R\frac{dy'}{dt}\right),$$

where P, Q, R are the angular velocities of (x', y', z') along the axes x', y', z' referred to the fixed axes; that is,

$$P=\alpha_3\frac{d\alpha_2}{dt}+\beta_3\frac{d\beta_2}{dt}+\gamma_3\frac{d\gamma_2}{dt}, \text{ etc.}$$

Since the axes of the ellipsoid are supposed to remain invariable, being of the form

$$V=K-Ax'^2-By'^2-Cz'^2$$

with constant K, A, B, C, we see that

$$\frac{\partial V}{\partial x'}-\frac{1}{\rho}\frac{\partial p}{\partial x'}, \text{ etc.,}$$

can be written

$$\frac{\partial V}{\partial x'}=-2Ax'=-2A(L\xi+M\eta+N\zeta);$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x'}=-2\sigma\left(\frac{\xi}{a^2}\frac{\partial\xi}{\partial x'}+\frac{\eta}{b^2}\frac{\partial\eta}{\partial x'}+\frac{\zeta}{c^2}\frac{\partial\zeta}{\partial x'}\right)=-2\sigma\left(\frac{\alpha\xi}{a^2}+\frac{\beta\eta}{ab}+\frac{\gamma\zeta}{ac}\right).$$

Put

$$\varphi=\sum\alpha''\frac{d\alpha'}{dt}, \qquad \psi=\sum\alpha\frac{d\alpha''}{dt}, \qquad \chi=\sum\alpha'\frac{d\alpha}{dt}.$$

Suppose that the external surface rotates with angular velocity ω around its figure-axis. Let it be the c-axis; then $P=Q=0$, $R\neq 0$, $A_{11}=A_{22}=-\omega^2$, $A_{12}=-dR/dt$, $A_{21}=dR/dt$, and other A_{rs} are zero. We obtain

$$\left.\begin{aligned}
&a\left(\psi\varphi+\frac{d\chi}{dt}\right)-b\,\frac{dR}{dt}=0, \qquad && a\left(\chi\varphi-\frac{d\psi}{dt}\right)-2bR\varphi=0,\\
&b\left(\psi\varphi-\frac{d\chi}{dt}\right)+a\,\frac{dR}{dt}=0, && b\left(\chi\psi+\frac{d\varphi}{dt}\right)-2aR\psi=0,\\
&c\left(\varphi\chi+\frac{d\psi}{dt}\right)=0, && c\left(\chi\psi-\frac{d\varphi}{dt}\right)=0.
\end{aligned}\right\}$$

These equations are satisfied in three cases:

$$I\begin{cases}\varphi=0,\\ \psi=0;\end{cases}\qquad \text{II}\begin{cases}\varphi=0,\\ \chi=R\dfrac{a}{b},\end{cases}\qquad \text{III}\begin{cases}\psi=0,\\ \chi=\dfrac{b}{a}R.\end{cases}$$

Suppose at first that $a\neq b$. We obtain at first $d\chi/dt=0$, $dR/dt=0$ for any of the three cases; and

$$\begin{aligned}
x'&=\xi\cos\chi t+\frac{a}{b}\eta\sin\chi t, \qquad & x&=x'\cos Rt-y'\sin Rt,\\
y'&=-\frac{b}{a}\xi\sin\chi t+\eta\cos\chi t, & y&=x'\sin Rt+y'\cos Rt,\\
z'&=\zeta; & z&=z'.
\end{aligned}$$

The axes $(x'\ y'\ z')$ rotate uniformly around the z-axis.

Let

$$\frac{c^2}{a^2}=s, \qquad \frac{c^2}{b^2}=t;$$

then the condition becomes

$$\frac{\chi^2+R^2}{2\pi\rho}=\varphi(s,t), \qquad \frac{2R\chi}{2\pi\rho}=-\sqrt{st}\,\psi(s,t),$$

where

$$\psi(s,t)=\int_0^\infty\left[\frac{1}{(1+s\kappa)(1+t\kappa)}-\frac{1}{1+\kappa}\right]\frac{d\kappa}{\Delta},$$

$$\varphi(s,t)=st\int_0^\infty\frac{\kappa d\kappa}{(1+s\kappa)(1+t\kappa)\Delta},$$

$$\Delta=\sqrt{(1+s\kappa)(1+t\kappa)(1+\kappa)};$$

Equation $\chi=0$ gives Jacobi's rigid motion; equation $R=0$ gives Dirichlet's internal harmonic motion (1860), and the external shape is fixed (Dedekind's ellipsoid, 1860).

The condition for $R\neq 0$, $\chi\neq 0$ can be transformed into

$$(\chi^2+\omega^2)^2-4\omega^2\chi^2=\frac{st}{t-s}\,(\bar{P}^2s-\bar{Q}^2t),$$

where

$$\bar{P}=2\pi\rho(1-s)\int_0^\infty \frac{\kappa(1+t\kappa)d\kappa}{\Delta^3}>0,$$

and

$$\bar{Q}=2\pi\rho(1-t)\int_0^\infty \frac{\kappa(1+s\kappa)}{\Delta^3}\,d\kappa>0.$$

The condition is satisfied if

$$\bar{P}\sqrt{s}>\bar{Q}\sqrt{t},\qquad \text{or}\qquad \sqrt{s}-s^{3/2}>\sqrt{t}-t^{3/2};$$

that is, s, t should be contained in the interval between 1/3 and t.

Next suppose that $a=b$. We have $Q=R=0, P\neq 0, a=b\neq c$, and $\psi=\chi=0, P=\text{const.}$, and $\varphi=\text{const.}$ A uniform rigid rotation occurs around the a-axis, and a homographic deformation occurs on the plane perpendicular to this axis. Contrary to the case $a\neq b$, the rigid-body rotation and the harmonic motion have a common axis which is one of the equatorial diameter.

Stekloff (1905, 1906, 1908, 1909) classified two cases $a>c$, $a=b$ and $a<c$, $a=b$. The first one is Dirichlet's case. In the second case, there are three solutions; one is Dirichlet's and the other two are new. Denote by ϵ' the positive root of

$$\log\frac{\epsilon+1}{\epsilon-1}=2\,\frac{15\epsilon^2-3\epsilon-4}{15\epsilon^3-3\epsilon^2-9\epsilon+1}.$$

There is only one positive root in the interval (1, 5/3). If $\omega^2/\pi>\Psi(\epsilon')$, where

$$\Psi(\epsilon)=\epsilon(\epsilon+1)(\epsilon^2-1)\left(\frac{3\epsilon^2-1}{2\epsilon}\log\frac{\epsilon+1}{\epsilon-1}-3\right),$$

then the solution is possible for $\omega^2/\pi>\Psi(\epsilon')$. If $\omega^2/\pi<\Psi(\epsilon')$, then there is one solution for $0<\omega^2/\pi<4/15$ and two solutions for $4/15<\omega^2/\pi<\Psi(\epsilon')$. Thus it is possible to have three solutions.

OSCILLATION OF AN ELLIPSOID

The stability character may be discussed by imposing small oscillations on the equilibrium figure, and studying the subsequent motion. It is the method of judging the stability by the characteristic exponents of Poincaré. An equilibrium figure is stable if the varied motion is a periodic oscillation around the equilibrium figure.

Appell (1920, 1921a) and Cartan (1922) considered small oscillations about the equilibrium state of a rotating fluid mass in general:

$$\left.\begin{aligned}\frac{\partial u}{\partial t}-2\omega v&=\frac{\partial\psi}{\partial x},\\ \frac{\partial v}{\partial t}+2\omega u&=\frac{\partial\psi}{\partial y},\\ \frac{\partial w}{\partial t}\qquad&=\frac{\partial\psi}{\partial z},\end{aligned}\right\}\qquad \begin{aligned}&\psi=V-\frac{p}{\rho}+\frac{1}{2}\,\omega^2(x^2+y^2),\\ &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.\end{aligned}$$

We obtain from these equations a partial differential equation

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi+4\omega^2\frac{\partial^2\psi}{\partial z^2}=0.$$

Putting $\delta x=e^{i\lambda t}\xi$, etc., $\psi=e^{i\lambda t}\psi_1$, $u=d\delta x/dt$, etc., we obtain

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial y^2}+\left(1-\frac{4\omega^2}{\lambda^2}\right)\frac{\partial^2\psi_1}{\partial z^2}=0.$$

Poincaré (1896, 1903, 1910a, 1910b) discussed the tidal motion over the oceans on the basis of the theory of integral equations (see Appendix A; see also Bertrand, 1923). Bryan (1890) and especially Hough (1897) considered the tidal theory on the basis of harmonic analysis. Proudman (1913, 1914, 1916, 1924, 1928, 1932, 1933) discussed the tidal theory on the basis of the theory of quadratic forms of an infinite number of variables by going back from Fredholm's theory of integral equations as in Poincaré's discussion to Hilbert's theory of integral equations. Goldsbrough (1928, 1929, 1930, 1931, 1933) discussed the tidal oscillations as the periodic solutions of differential equations.

A complete discussion of ellipsoid-type oscillations of a Maclaurin spheroid has been published by Hargreaves (1914). Such oscillations (Riemann, 1860) were shown to have a period just one-half the rotation period at the bifurcation point of the Jacobi ellipsoid series and the Maclaurin spheroid series. The oscillations of a spheroid are divided into polar and equatorial. In a polar oscillation, the equator is always circular, but the equatorial radius and polar radius are subject to periodic change. In an equatorial oscillation, the polar axis is invariable, but the equator is subject to periodic elliptic deformation. The mode of oscillations in the beginning of the Jacobi linear series is not much different from that of a spheroid near the bifurcation point. As we proceed farther from the spheroid, the distinction between polar and equatorial becomes unsuitable. If the word *polar* is applied to the long axis, and the word *equator* to the nearly circular ellipse containing the short axis, then the oscillation may be classified as polar and equatorial. In the beginning of the Jacobi linear series, the equatorial oscillation has just one-half the period of rotation, and the polar oscillation has a shorter period. As the ellipsoid becomes slightly elongated, the two periods gradually diverge; the first period increases, and the second decreases. In the limit of extreme elongation, the first period becomes equal to the rotation period, and the second tends to a value as the rotation period tends to infinity. The two frequencies n_e and n_p are finite for small oblateness, and n_e, which is the greater at first, decreases rapidly with increasing oblateness and $n_e=n_p$ for $c/a=0.5892$ when $n^2/\omega^2=4.166$ . . . . The bifurcation occurs at $c/a=0.5827$, $n_e=2\omega$, $n_p^2/\omega^2=4.1182$; $n_p=2\omega$ is reached for $c/a=0.5612$.

The amplitude ratios $da{:}db{:}dc$ of the oscillation of frequency n are determined by (Riemann, 1861)

$$[a^2E_{aa}-n^2(a^2+c^2)]\frac{da}{a}+(abE_{ab}-n^2c^2)\frac{db}{b}=0,$$

$$[b^2E_{bb}-n^2(b^2+c^2)]\frac{db}{b}+(abE_{ab}-n^2c^2)\frac{da}{a}=0;\qquad abc=\text{constant}.$$

The frequency n^2 is given by the quadratic equation for n^2 by eliminating the ratio da/db from these equations; the total energy is expressed by $mE/5$, and E_{aa}, etc., are the second derivatives of E with regard to a and b, respectively.

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CHAPTER III

Lamé Functions

LINEAR DIFFERENTIAL EQUATIONS

Consider the differential equation

$$\frac{d^2u}{dz^2}+p(z)\frac{du}{dz}+q(z)u=0,$$

where $p(z)$, $q(z)$ are analytic but with a finite number of poles. A point at which $p(z)$, $q(z)$ are both analytic is called *ordinary*; otherwise it is *singular*. If, although $p(z)$ or $q(z)$ or both may have poles at $z=c$, $(z-c)p(z)$ and $(z-c)^2q(z)$ are analytic at $z=c$, then such a point $z=c$ is called *regular*; otherwise the point is called *irregular*. This condition is not only necessary but sufficient (Ince, 1927).

Write

$$(z-c)^2\frac{d^2u}{dz^2}+(z-c)P(z-c)\frac{du}{dz}+Q(z-c)u=0,$$

and

$$P(z-c)=p_0+p_1(z-c)+p_2(z-c)^2+\ldots,$$

$$Q(z-c)=q_0+q_1(z-c)+q_2(z-c)^2+\ldots,$$

and try the solution

$$u=(z-c)^\alpha\left[1+\sum_{n=1}^{\infty}a_n(z-c)^n\right].$$

We obtain

$$\alpha^2+(p_0-1)\alpha+q_0=0,$$

$$a_1[(\alpha+1)^2+(p_0-1)(\alpha+1)+q_0]+\alpha p_1+q_1=0,\ \ldots.$$

The equation $F(\alpha)\equiv\alpha^2+(p_0-1)\alpha+q_0=0$ is called the *indicial equation*. The other co-

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efficients a_1, a_2, . . . are determined successively after we determine the roots $\alpha=\rho_1$, ρ_2 by this equation.

$$w_1(z)=(z-c)^{\rho_1}\left[1+\sum_{n=1}^{\infty} a_n(z-c)^n\right], \qquad w_2(z)=(z-c)^{\rho_2}\left[1+\sum_{n=1}^{\infty} a_n'(z-c)^n\right].$$

If the roots are ρ and $\rho+n(n=0, 1, 2 \ldots)$, then this formula for $w_1(z)$, $w_2(z)$ fails, and $w_2(z)$ is written

$$Aw_1(z)+B\left[w_1(z)\log(z-c)+(z-c)^{\rho_2}\sum_{n=1}^{\infty} h_n(z-c)^n\right]$$

with arbitrary constants A, B. The coefficients h_n are determined by a process similar to the above. The behavior at infinity can be examined by putting $z=1/z_1$. The point $z=\infty$ is regular or irregular accordingly as $z_1=0$ is regular or irregular. If $zp(z)$, $z^2q(z)$ are analytic, then $z=\infty$ is regular.

A linear differential equation with only regular singularities in the whole domain including $z=\infty$ is said to be *Fuchsian.* When two singularities coincide, we call it a *confluence.* Suppose that a_1, a_2, a_3, a_4, ∞ are the only regular singular points and other points are ordinary. Let the exponents be α_r, $\beta_r(r=1, 2, 3$, and $4)$, and the exponents at ∞ be μ_1, μ_2. The equation is of the form

$$\frac{d^2u}{dz^2}+\left(\sum_{r=1}^{4}\frac{1-\alpha_r-\beta_r}{z-a_r}\right)\frac{du}{dz}+\left[\sum_{r=1}^{4}\frac{\alpha_r\beta_r}{(z-a_r)^2}+\frac{Az^2+2Bz+C}{\prod_{r=1}^{4}(z-a_r)}\right]u=0,$$

where A is such that μ_1, μ_2 are the roots of

$$\mu^2+\mu\left[\sum_{r=1}^{4}(\alpha_r+\beta_r)-3\right]+\sum_{r=1}^{4}\alpha_r\beta_r+A=0,$$

and B, C are constants. Klein (1894) and Bôcher (1894) have shown that linear differential equations in mathematical physics can be expressed in a form such that the difference of the two exponents is equal to 1/2 (*elementary singularity*). If two regular singularities coalesce, we have a regular singularity with arbitrary exponents. If three or more than three regular singularities coalesce, we have an irregular singularity. A linear differential equation with rational coefficients has a definite number of regular singularities and a definite number of irregular singularities. Thus with $\beta_r=\alpha_r+1/2$ $(r=1, 2, 3, 4)$ we have

$$\frac{d^2u}{dz^2}+\left(\sum_{r=1}^{4}\frac{\frac{1}{2}-2\alpha_r}{z-a_r}\right)\frac{du}{dz}+\left[\sum_{r=1}^{4}\frac{\alpha_r\left(\alpha_r+\frac{1}{2}\right)}{(z-a_r)^2}+\frac{Az^2+2Bz+C}{\prod_{r=1}^{4}(z-a_r)}\right]u=0.$$

For $z=\infty$ to be an elementary singularity, we should have

$$\mu_1-\mu_2=\frac{1}{2} \quad\text{and}\quad A=\left(\sum_{r=1}^{4}\alpha_r\right)^2-\sum_{r=1}^{4}\alpha_r^2-\frac{3}{2}\sum_{r=1}^{4}\alpha_r+\frac{3}{16}.$$

Thus two constants B and C are arbitrary. By a further transformation

$$u=v\prod_{r=1}^{4}(z-a_r)^{\alpha_r},$$

we can reduce the equation to

$$\frac{d^2v}{dz^2}+\left(\sum_{r=1}^{4}\frac{1/2}{z-a_r}\right)\frac{dv}{dz}+\frac{C+2Bz+Az^2}{\prod_{r=1}^{4}(z-a_r)}v=0,$$

with $A=3/16$. This is the *generalized Lamé equation.* The most general type is

$$\frac{d^2v}{dz^2}+\frac{1}{2}\left(\sum_{r=1}^{r}\frac{1}{z-a_r}\right)\frac{dv}{dz}+\frac{1}{(z-a_1)\ .\ .\ .\ (z-a_r)}\left[\frac{r(r-4)}{16}z^{r-2}+c_{r-3}z^{r-3}+\ .\ .\ .\ +c_0\right]v=0.$$

This is called the *Lamé-Klein equation*, which is the basis of the theory of *cyclids* (Bôcher, 1894). Thus a linear differential equation is specified by the singular points and their exponents. According to Riemann, the generalized Lamé equation is written

$$P\left\{\begin{matrix} a_1 & a_2 & a_3 & a_4 & \infty \\ 0 & 0 & 0 & 0 & z \\ 1/2 & 1/2 & 1/2 & 1/2 & \end{matrix}\right\},$$

and specified by three elementary singularities and one nonelementary singularity without any essential singularity.

Put $a_4\to 0$, $\lim C/a_r=h/4$, $\lim 2B/a_4=n(n+1)/4$ in the generalized Lamé equation; this gives the *Lamé equation*

$$\frac{d^2w}{dz^2}+\left(\frac{1/2}{z-a_1}+\frac{1/2}{z-a_2}+\frac{1/2}{z-a_3}\right)\frac{dw}{dz}-\frac{h+n(n+1)z}{4(z-a_1)(z-a_2)(z-a_3)}w=0.$$

The irreducible constants are $(a_3+a_2)/(a_3-a_1)$, h, and n. Put $a_2\to a_3\to 1$, $a_1\to 0$; then we have the associated *Legendre equation*

$$(1-x^2)\frac{d^2w}{dx^2}-2x\frac{dw}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]w=0.$$

Put $a_1\to 0$, $a_2\to 1$, $a_3\to a_4\to\infty$ and let $\lim\,[C/(a_3a_4)]=a/4$, $\lim\,[2B/(a_3a_4)]=k^2/4$; then we get the *Mathieu equation*

$$\frac{d^2w}{dz^2}+\left(\frac{1/2}{z}+\frac{1/2}{z-1}\right)\frac{dw}{dz}-\frac{a+k^2z}{4z(z-1)}w=0,$$

or

$$\frac{d^2w}{dx^2}+(a+k^2\cos^2 x)w=0, \qquad z=\cos^2 x.$$

Put $a_1 \to a_2 \to 0$, $a_3 \to a_4 \to \infty$, and let $\lim C/(a_3 a_4) = n^2/4$, $\lim 2B/(a_3 a_4) = k^2/4$; then we get the *Bessel equation*

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \frac{z-n^2}{4z^2}\,w = 0,$$

or

$$\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dw} + \left(1 - \frac{n^2}{x^2}\right)w = 0, \qquad z = x^2.$$

ELLIPTIC COORDINATES

Consider

$$\frac{x^2}{\lambda^2 - a^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} - 1 = 0, \tag{41}$$

$$c < b < a, \qquad \lambda^2 - a^2 < \lambda^2 - b^2 < \lambda^2 - c^2.$$

The surfaces obtained by varying λ are *confocal.* We obtain three surfaces corresponding to the three roots $\lambda^2 : c < \nu < b < \mu < a < \rho$. For $\rho > a$, it is an ellipsoid; for $b < \mu < a$, a hyperboloid of one sheet; for $c < \nu < b$, a hyperboloid of two sheets; ρ, μ, ν are called the *elliptic coordinates* of a point in space. For a given set ρ, μ, ν, there are eight points located symmetrically with regard to the three coordinate planes; for a given set of x, y, z, there is only one set ρ, μ, ν. We have identically

$$\frac{x^2}{\lambda^2 - a^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} - 1 = \frac{(\rho^2 - \lambda^2)(\mu^2 - \lambda^2)(\nu^2 - \lambda^2)}{(\lambda^2 - a^2)(\lambda^2 - b^2)(\lambda^2 - c^2)}. \tag{42}$$

Multiply both sides by $\lambda^2 - a^2$ and put $\lambda^2 = a^2$; then we have

$$x^2 = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{(a^2 - b^2)(a^2 - c^2)};$$

similarly

$$\left.\begin{aligned} y^2 &= \frac{(\rho^2 - b^2)(\mu^2 - b^2)(\nu^2 - b^2)}{(b^2 - a^2)(b^2 - c^2)}, \\ z^2 &= \frac{(\rho^2 - c^2)(\mu^2 - c^2)(\nu^2 - c^2)}{(c^2 - a^2)(c^2 - b^2)}. \end{aligned}\right\} \tag{43}$$

The elliptic coordinates ρ, μ, ν are orthogonal:

$$\left.\begin{aligned} &ds^2 = dx^2 + dy^2 + dz^2 = \alpha^2 d\rho^2 + \beta^2 d\mu^2 + \gamma^2 d\nu^2, \\ &\alpha = \frac{Q}{A\sqrt{\mu^2 - \nu^2}}, \qquad \beta = \frac{Q}{B\sqrt{\nu^2 - \rho^2}}, \qquad \gamma = \frac{Q}{C\sqrt{\rho^2 - \mu^2}}, \end{aligned}\right\} \tag{44}$$

$$\left.\begin{aligned}
Q^2 &= (\mu^2-\rho^2)\,(\rho^2-\nu^2)\,(\nu^2-\mu^2),\\
A^2 &= \frac{(\rho^2-a^2)\,(\rho^2-b^2)\,(\rho^2-c^2)}{\rho^2},\\
B^2 &= \frac{(\mu^2-a^2)\,(\mu^2-b^2)\,(\mu^2-c^2)}{\mu^2},\\
C^2 &= \frac{(\nu^2-a^2)\,(\nu^2-b^2)\,(\nu^2-c^2)}{\nu^2}.
\end{aligned}\right\} \tag{45}$$

Here $A^2>0$, $B^2>0$, $C^2<0$. Hence A, C are real, and B pure imaginary. Thus the Laplace equation takes the form

$$\Delta V = \frac{1}{\alpha\beta\gamma}\sum \frac{\partial}{\partial\rho}\left[\frac{\beta\gamma}{\alpha}\frac{\partial V}{\partial\rho}\right]=0,$$

or

$$\sum A\;(\mu^2-\nu^2)\,\frac{\partial}{\partial\rho}\left(A\,\frac{\partial V}{\partial\rho}\right)=0. \tag{46}$$

Put

$$\frac{d\rho}{A}=du, \qquad \frac{d\mu}{B}=dv, \qquad \frac{d\nu}{C}=dw; \tag{47}$$

then

$$(\mu^2-\nu^2)\,\frac{\partial^2V}{\partial u^2}+(\nu^2-\rho^2)\,\frac{\partial^2V}{\partial v^2}+(\rho^2-\mu^2)\,\frac{\partial^2V}{\partial w^2}=0. \tag{48}$$

Now write

$$\frac{ds}{\sqrt{4s^3-g_2s-g_3}}=dz, \qquad s=\wp\, z, \tag{49}$$

with invariants g_2 and g_3. Assume that $s=\infty$; that is, $\wp z=\infty$ is a pole of order 2 at $z=0$:

$$\wp z=\frac{1}{z^2}+C_1z^2+\ldots;$$

$\wp\, z$ has two periods 2ω and $2\omega'$, and the three roots are real:

$$\left.\begin{aligned}
&4s^3-g_2s-g_3\equiv 4(s-e_1)\,(s-e_2)\,(s-e_3),\\
&e_1+e_2+e_3=0, \quad e_3<e_2<e_1.
\end{aligned}\right\} \tag{50}$$

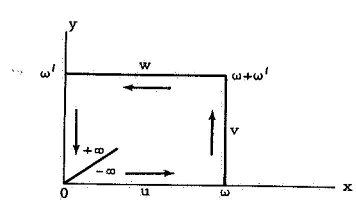
Suppose that $g_2<0$; then ω is real, and ω' is imaginary:

$z=0\;.\;.\;.\;\omega \quad : s=\wp\, z=+\infty\;.\;.\;.\;e_1$

$z=\omega\;.\;.\;.\;\omega+\omega': \quad \wp\, z=e_1\;.\;.\;.\;e_2$

$z=\omega+\omega'\;.\;.\;.\;\omega': \quad \wp\, z=e_2\;.\;.\;.\;e_3$

$z=\omega'\;.\;.\;.\;0 \quad : \quad \wp\, z=e_3\;.\;.\;.\;-\infty.$



As z describes a path $0 \to \omega \to \omega+\omega' \to \omega' \to 0$, the variable z always decreases;

$$\frac{ds}{dz} = \wp' z = \sqrt{4s^3 - g_2 s - g_3} = 2\sqrt{(\wp z - e_1)(\wp z - e_2)(\wp z - e_3)}; \tag{51}$$

$$\left.\begin{aligned} &2\wp'\wp'' = 12\wp^2\wp' - g_2\wp', \qquad \wp'' = 6\wp^2 - g_2/2; \\ &\wp(0) = \pm\infty, \qquad \wp\omega = e_1, \qquad \wp(\omega+\omega') = e_2, \qquad \wp\omega' = e_3. \end{aligned}\right. \tag{52}$$

Put

$$\sqrt{\wp - e_1} = \wp_1, \qquad \sqrt{\wp - e_2} = \wp_2, \qquad \sqrt{\wp - e_3} = \wp_3; \tag{53}$$

$\wp_1$, $\wp_2$, $\wp_3$ are doubly periodic but the periods are different:

	2ω	$2\omega'$	period	
$\wp z$	$+$	$+$	2ω	$2\omega'$
$\wp_1 z$	$+$	$-$	2ω	$4\omega'$
$\wp_2 z$	$-$	$-$	4ω	$4\omega'$
$\wp_3 z$	$-$	$+$	4ω	$2\omega'$

(54)

Now

$$c^2 < \nu^2 < b^2 < \mu^2 < a^2 < \rho^2.$$

Put $\rho^2 = s + h$, $a^2 - h = e_1$, $b^2 - h = e_2$, $c^2 - h = e_3$, $h = (1/3)(a^2 + b^2 + c^2)$; then we have

$$\frac{ds}{2\sqrt{(s-e_1)(s-e_2)(s-e_3)}} = du; \tag{55}$$

u varies on the real axis from 0 to ω, and s varies from $+\infty$ to $a^2 - h = e_1$. Similarly, put $\mu^2 = \wp v + h$, and let v vary from ω to $\omega+\omega'$ so that μ^2 is real. Put $\nu^2 = \wp w + h$ and let w vary from $\omega+\omega'$ to ω'. We have

$$\left.\begin{aligned} x^2 &= \frac{(\wp u - e_1)(\wp v - e_1)(\wp w - e_1)}{(e_1 - e_2)(e_1 - e_3)}, \\ y^2 &= \frac{(\wp u - e_2)(\wp v - e_2)(\wp w - e_2)}{(e_2 - e_3)(e_2 - e_1)}, \\ z^2 &= \frac{(\wp u - e_3)(\wp v - e_3)(\wp w - e_3)}{(e_3 - e_1)(e_3 - e_2)}; \end{aligned}\right\} \tag{56}$$

or, with

$$\sqrt{\wp z - e_r} = \frac{\sigma_r(z)}{\sigma(z)}, \qquad \frac{d^2}{dz^2}\log\sigma(z) = \frac{d}{dz}\zeta(z) = -\wp(z),$$

$$\omega_1 = \omega, \qquad \omega_2 = \omega', \qquad \omega_3 = -\omega - \omega', \qquad \eta_r = \zeta(\omega_r), \qquad (r = 1, 2, 3),$$

we have

$$x=\pm e^{-\eta_1\omega_1}\sigma^2(\omega_1)\frac{\sigma_1(u)\sigma_1(v)\sigma_1(w)}{\sigma(u)\sigma(v)\sigma(w)},$$

$$y=\pm e^{-\eta_2\omega_2}\sigma^2(\omega_2)\frac{\sigma_2(u)\sigma_2(v)\sigma_2(w)}{\sigma(u)\sigma(v)\sigma(w)},$$

and

$$z=\pm e^{-\eta_3\omega_3}\sigma^2(\omega_3)\frac{\sigma_3(u)\sigma_3(v)\sigma_3(w)}{\sigma(u)\sigma(v)\sigma(w)},$$

where

$$\sigma_r(u)=e^{-\eta_r z}\frac{\sigma(z+\omega_r)}{\sigma(\omega_r)} \qquad (r=1, 2, 3).$$

The Laplace equation now takes the form

$$(\wp v-\wp w)\frac{\partial^2 V}{\partial u^2}+(\wp w-\wp u)\frac{\partial^2 V}{\partial v^2}+(\wp u-\wp v)\frac{\partial^2 V}{\partial w^2}=0. \tag{57}$$

ELLIPSOIDAL HARMONICS

We now express ρ, μ, ν defined by equation 43 in the form of polynomials of x, y, z. Consider a polynomial $Q_n(x, y, z)$ of degree n and replace x, y, z by their expressions (equation 43) in terms of ρ, μ, ν. Quantity Q_n is symmetric with regard to ρ, μ, ν since it is symmetric with regard to x, y, z. If we consider $\sqrt{\rho^2-a^2}$ as of degree 1 with regard to ρ, then Q_n is of degree n with regard to ρ. We distinguish four classes:

1st class:

$$Q_n=P(x^2, y^2, z^2), \qquad n=2k.$$

$$Q(x^2, y^2, z^2)=\Phi(\rho^2, \mu^2, \nu^2).$$

This is of degree $2k$ with regard to ρ, μ, ν.

2nd class:

$$Q_n=xP(x^2, y^2, z^2), \qquad n=2k+1.$$

$$xP(x^2, y^2, z^2)= C\sqrt{\rho^2-a^2}\sqrt{\mu^2-a^2}\sqrt{\nu^2-a^2}\,\Phi(\rho^2, \mu^2, \nu^2),$$

$$yP(x^2, y^2, z^2)=C'\sqrt{\rho^2-b^2}\sqrt{\mu^2-b^2}\sqrt{\nu^2-b^2}\,\Phi(\rho^2, \mu^2, \nu^2),$$

$$zP(x^2, y^2, z^2)=C''\sqrt{\rho^2-c^2}\sqrt{\mu^2-c^2}\sqrt{\nu^2-c^2}\,\Phi(\rho^2, \mu^2, \nu^2).$$

3rd class:

$$Q_n = yzP(x^2, y^2, z^2), \qquad n = 2k+2,$$

$$yzP(x^2, y^2, z^2) \\ = C_1\sqrt{(\rho^2-b^2)(\rho^2-c^2)}\ \sqrt{(\mu^2-b^2)(\mu^2-c^2)}\ \sqrt{(\nu^2-b^2)(\nu^2-c^2)}\ \Phi(\rho^2, \mu^2, \nu^2),$$

$$zxP(x^2, y^2, z^2) \\ = C_1'\sqrt{(\rho^2-c^2)(\rho^2-a^2)}\ \sqrt{(\mu^2-c^2)(\mu^2-a^2)}\ \sqrt{(\nu^2-c^2)(\nu^2-a^2)}\ \Phi(\rho^2, \mu^2, \nu^2),$$

$$xyP(x^2, y^2, z^2) \\ = C_1''\sqrt{(\rho^2-a^2)(\rho^2-b^2)}\ \sqrt{(\mu^2-a^2)(\mu^2-b^2)}\ \sqrt{(\nu^2-a^2)(\nu^2-b^2)}\ \Phi(\rho^2, \mu^2, \nu^2).$$

4th class:

$$Q_n = xyzP(x^2, y^2, z^2), \qquad n = 2k+3,$$

$$xyzP(x^2, y^2, z^2) = C_2 \prod_{\rho,\mu,\nu} \sqrt{(\rho^2-a^2)(\rho^2-b^2)(\rho^2-c^2)}\Phi(\rho^2, \mu^2, \nu^2).$$

Given an arbitrary polynomial P_n, we can express it as a sum of eight expressions of each of these eight forms; it is symmetric with regard to ρ, μ, ν and of the same degree as P_n with regard to x, y, z.

Next, consider the reverse process.

(1) Take a symmetric function with regard to ρ, μ, ν such as $f(\rho^2)f(\mu^2)f(\nu^2)$. Since this is symmetric with regard to the roots of

$$\frac{x^2}{\lambda^2-a^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}-1=0,$$

it should be rational with regard to the coefficients of this equation; hence it should be rational with regard to x^2, y^2, z^2. In fact, by factorizing $f(\rho^2)$ in the form

$$f(\rho^2) = (\rho^2-\alpha_1)(\rho^2-\alpha_2)\ \ldots\ (\rho^2-\alpha_k),$$

we obtain

$$f(\rho^2)f(\mu^2)f(\nu^2) = \prod_{i=1}^{k} [(\rho^2-\alpha_i)(\mu^2-\alpha_i)(\nu^2-\alpha_i)].$$

Put $\lambda^2 = \alpha_i$ in our fundamental identity (equation 42); we obtain

$$(\rho^2-\alpha_i)\ (\mu^2-\alpha_i)\ (\nu^2-\alpha_i) = C\left(\frac{x^2}{\alpha_i-a^2}+\frac{y^2}{\alpha_i-b^2}+\frac{z^2}{\alpha_i-c^2}-1\right), \tag{58}$$

or

$$f(\rho^2)f(\mu^2)f(\nu^2) = C'\prod_{i=1}^{k}\left(\frac{x^2}{\alpha_i-a^2}+\frac{y^2}{\alpha_i-b^2}+\frac{z^2}{\alpha_i-c^2}-1\right) = Q(x^2, y^2, z^2).$$

(2) Similarly.

$$\sqrt{\rho^2-a^2}\,f(\rho^2)\ \sqrt{\mu^2-a^2}\,f(\mu^2)\ \sqrt{\nu^2-a^2}\,f(\nu^2) = xQ\,(x^2, y^2, z^2),$$

$$\sqrt{\rho^2-b^2}\,f(\rho^2)\ \sqrt{\mu^2-b^2}\,f(\mu^2)\ \sqrt{\nu^2-b^2}\,f(\nu^2) = yQ(x^2, y^2, z^2),$$

$$\sqrt{\rho^2-c^2}\,f(\rho^2)\ \sqrt{\mu^2-c^2}\,f(\mu^2)\ \sqrt{\nu^2-c^2}\,f(\nu^2) = zQ(x^2, y^2, z^2).$$

(3)
$$\prod_{\rho,\,\mu,\,\nu} \sqrt{(\rho^2-a^2)\,(\rho^2-b^2)}\,f(\rho^2) = xyQ(x^2, y^2, z^2),$$

$$\prod_{\rho,\,\mu,\,\nu} \sqrt{(\rho^2-b^2)\,(\rho^2-c^2)}\,f(\rho^2) = yzQ(x^2, y^2, z^2),$$

$$\prod_{\rho,\,\mu,\,\nu} \sqrt{(\rho^2-c^2)\,(\rho^2-a^2)}\,f(\rho^2) = zxQ(x^2, y^2, z^2).$$

(4)
$$\prod_{\rho,\,\mu,\,\nu} \sqrt{(\rho^2-a^2)\,(\rho^2-b^2)\,(\rho^2-c^2)}\,f(\rho^2) = xyzQ(x^2, y^2, z^2).$$

Problems concerning a sphere are dealt with by a linear combination of spherical harmonics

$$r^nP_n(\cos\theta), \qquad r^nP_n^m\,(\cos\theta)\,{}^{\cos}_{\sin}\,m\varphi,$$

with positive integers m, n. If we put $\cos\theta = z/r$, then r^nP_n $(\cos\theta)$ is expressed as a product of linear factors with respect to x^2, y^2, z^2. Tesseral harmonics are either of the eight forms, such as a product of linear factors with regard to x^2, y^2, z^2, multiplied by either 1, x, y, z, yz, zx, xy, xyz.

Similar problems concerning an ellipsoid are dealt with by ellipsoidal harmonics (Lamé, 1839; Niven, 1892; Hobson, 1931).

$$\frac{x^2}{\theta_\alpha-a^2}+\frac{y^2}{\theta_\alpha-b^2}+\frac{z^2}{\theta_\alpha-c^2}-1 \equiv \Theta_\alpha, \qquad \theta_\alpha = \lambda_\alpha^2; \tag{59}$$

then the foregoing four types of expressions are written

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\ \Theta_1\Theta_2\ldots\Theta_m;$$

$\Theta_1\Theta_2\ldots\Theta_m = \Pi\,(\Theta)$ is an ellipsoidal harmonic of the first class; $x\Pi\,(\Theta)$ is of the second, $yz\Pi(\Theta)$ of the third, $xyz\Pi(\Theta)$ of the fourth.

In order that $\Pi(\Theta) = \Theta_1\Theta_2\ldots\Theta_m$ may satisfy the Laplace equation, it should satisfy

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Pi\,(\Theta) = \sum_{p=1}^{m}\frac{\partial\Pi(\Theta)}{\partial\Theta_p}\left(\frac{2}{\theta_p-a^2}+\frac{2}{\theta_p-b^2}+\frac{2}{\theta_p-c^2}+\sum_{q=1}^{m}{}'\,\frac{8}{\theta_p-\theta_q}\right) = 0, \tag{60}$$

or we would have

$$\frac{1}{\theta_p-a^2}+\frac{1}{\theta_p-b^2}+\frac{1}{\theta_p-c^2}+\sum_q^{m}{}'\frac{4}{\theta_p-\theta_q}=0 \qquad (p=1,2\ldots,m).$$

Put

$$\Lambda_1(\theta)=\prod_{q=1}^{m}(\theta-\theta_q),$$

a polynomial of degree m with regard to θ:

$$\frac{d\Lambda_1(\theta)}{d\theta}=\Lambda_1'(\theta)$$

is a sum of $m-1$ products among $\theta-\theta_1,\ldots,\theta-\theta_m$, and

$$\frac{d^2\Lambda_1(\theta)}{d\theta^2}=\Lambda_1''(\theta)$$

is a sum of $m-2$ products among $\theta-\theta_1,\ldots,\theta-\theta_m$. Hence

$$\frac{\Lambda_1''(\theta_p)}{\Lambda_1'(\theta_p)}$$

is twice the sum of the reciprocals of $\theta_p-\theta_1,\ldots,\theta_p-\theta_m$. Since

$$\prod_{p=1}^{m}(\Theta_p)$$

is harmonic,

$$\frac{1}{\theta-a^2}+\frac{1}{\theta-b^2}+\frac{1}{\theta-c^2}+\frac{2\Lambda_1''(\theta)}{\Lambda_1'(\theta)}$$

should vanish for $\theta=\theta_1,\ldots,\theta_m$. Hence

$$(\theta-a^2)(\theta-b^2)(\theta-c^2)\Lambda_1''(\theta)+\frac{1}{2}\left[\sum_{a,b,c}(\theta-a^2)(\theta-b^2)\right]\Lambda_1'(\theta)$$

is a polynomial in θ which vanishes for $\theta=\theta_1,\ldots,\theta_m$. Hence it should have factors $(\theta-\theta_1)\ldots(\theta-\theta_m)$ and be of degree $m+1$ with regard to θ; and indeed the coefficient of θ^{m-1} is $m(m+1/2)$. The factor of $[(\theta-\theta_1)\ldots(\theta-\theta_m)]$ is thus $m(m+1/2)\theta+C$, where C is determined later. Hence

$$(\theta-a^2)(\theta-b^2)(\theta-c^2)\Lambda_1''(\theta)+(1/2)\left[\sum_{a,b,c}(\theta-b^2)(\theta-c^2)\right]\Lambda_1'(\theta)$$
$$=[m(m+1/2)\theta+C/4]\Lambda_1(\theta).$$

An ellipsoidal harmonic of the first class of even degree n is of the form

$$\prod_{p=1}^{n/2}\left(\frac{x^2}{\theta_p-a^2}+\frac{y^2}{\theta_p-b^2}+\frac{z^2}{\theta_p-c^2}-1\right),$$

where $\theta_1, \ldots, \theta_{n/2}$ are zeros of a polynomial $\Lambda_1(\theta)$ of degree $n/2$, and is the solution of the differential equation

$$4\sqrt{(\theta-a^2)(\theta-b^2)(\theta-c^2)}\,\frac{d}{d\theta}\left[\sqrt{(\theta-a^2)(\theta-b^2)(\theta-c^2)}\,\frac{d\Lambda_1}{d\theta}\right] = [n(n+1)\theta+C]\Lambda_1(\theta). \tag{61}$$

This is called the *Lamé equation.*

For the second class, take $x\Pi(\Theta_p)$ of degree $2m+1$, and consider

$$\Lambda_2(\theta)=\prod_{q=1}^{m}(\theta-\theta_q).$$

Put

$$\Lambda_2(\theta)=\frac{\Lambda(\theta)}{\sqrt{\theta-a^2}};$$

we again obtain the Lamé equation

$$4\sqrt{(\theta-a^2)(\theta-b^2)(\theta-c^2)}\,\frac{d}{d\theta}\left[\sqrt{(\theta-a^2)(\theta-b^2)(\theta-c^2)}\,\frac{d\Lambda(\theta)}{d\theta}\right] = [(2m+1)(2m+2)\theta+C]\Lambda(\theta).$$

For the third class, take

$$yz\prod_{p=1}^{m}(\Theta_p)$$

of degree $2m+2$; we obtain the same equation with $n=2m+2$.

For the fourth class, take

$$xyz\prod_{p=1}^{m}(\Theta_p)$$

of degree $2m+3$; we obtain the same equation with $n=2m+3$.

As we shall see later, there are $n/2+1$ of the first class and $3n/2$ of the third class when n is even, and there are $3(n+1)/2$ of the second and $(n-1)/2$ of the fourth class when n is odd; in total there are $2n+1$ harmonics.

LAMÉ EQUATION

We try the solution of $\Delta V=0$ in the form $V=f(\rho^2)f(\mu^2)f(\nu^2)=RMN$; then, substituting in equation 46, we obtain

$$(\mu^2-\nu^2)\frac{A}{R}\frac{d}{d\rho}\left(A\frac{dR}{d\rho}\right)+(\nu^2-\rho^2)\frac{B}{M}\frac{d}{d\mu}\left(B\frac{dM}{d\mu}\right)+(\rho^2-\mu^2)\frac{C}{N}\frac{d}{d\nu}\left(C\frac{dN}{d\nu}\right)=0. \tag{62}$$

Obviously,

$$(\mu^2-\nu^2)+(\nu^2-\rho^2)+(\rho^2-\mu^2)\equiv 0,$$

$$(\mu^2-\nu^2)\rho^2+(\nu^2-\rho^2)\mu^2+(\rho^2-\mu^2)\nu^2\equiv 0.$$

Hence

$$\begin{vmatrix} \dfrac{A}{R}\dfrac{d}{d\rho}\left(A\dfrac{dR}{d\rho}\right) & \dfrac{B}{M}\dfrac{d}{d\mu}\left(B\dfrac{dM}{d\mu}\right) & \dfrac{C}{N}\dfrac{d}{d\nu}\left(C\dfrac{dN}{d\nu}\right) \\ \rho^2 & \mu^2 & \nu^2 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

or, with constants K and H, and by equation 47,

$$\frac{d^2R}{du^2} = (H\rho^2 + K)R, \qquad \frac{d^2M}{dv^2} = (H\mu^2 + K)M, \qquad \frac{d^2N}{dw^2} = (H\nu^2 + K)N. \tag{63}$$

Let us determine H and K so that R is a polynomial $f(\rho^2)$ of ρ multiplied by either of the eight factors 1, $\sqrt{\rho^2 - a^2}$, $\sqrt{\rho^2 - b^2}$, $\sqrt{\rho^2 - c^2}$, $\sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}$, $\sqrt{(\rho^2 - c^2)(\rho^2 - a^2)}$, $\sqrt{(\rho^2 - a^2)(\rho^2 - b^2)}$, $\sqrt{(\rho^2 - a^2)(\rho^2 - b^2)(\rho^2 - c^2)}$; $f(\rho^2)$ or such products are called *Lamé functions*, and the product $f(\rho^2)f(\mu^2)f(\nu^2)$ or similar products are called *Lamé products*.

A similar computation on equation 57 gives

$$\frac{d^2R}{du^2} - (H\wp u + B)R = 0, \qquad B = Hh + K. \tag{64}$$

This is called the *Lamé equation*. It can be shown that $H = n(n+1)$; but B or K should be so chosen that the solution is of the form we desire.

By a similar procedure we can derive from ellipsoidal harmonics II(Θ) the Lamé equation in the form

$$\frac{d^2R}{d(\rho^2)^2} + \left(\frac{1/2}{\rho^2 - a^2} + \frac{1/2}{\rho^2 - b^2} + \frac{1/2}{\rho^2 - c^2}\right)\frac{dR}{d(\rho^2)} = \frac{[n(n+1)\rho^2 + B']}{4(\rho^2 - a^2)(\rho^2 - b^2)(\rho^2 - c^2)} R.$$

This is called the *algebraic form* of the Lamé equation (Stieltjes, 1885; Klein, 1894; Bôcher, 1894); putting $B' = B - (1/3)n(n+1)(a^2 + b^2 + c^2)$, we obtain

$$\frac{d^2R}{du^2} - [n(n+1)\wp u + B]R = 0,$$

which is of Weierstrass's form. Also, by putting $s = \wp u$, we obtain

$$\frac{d^2R}{ds^2} + \left(\frac{1/2}{s - e_1} + \frac{1/2}{s - e_2} + \frac{1/2}{s - e_3}\right)\frac{dR}{ds} = \frac{[n(n+1)s + B]R}{4(s - e_1)(s - e_2)(s - e_3)},$$

which has the singularities at e_1, e_2, e_3; the corresponding exponents are 0 and 1/2. The exponents for the singularity at ∞ are $-n/2$ and $(n+1)1/2$. Put $\rho = k$ sn$x$, $a = k$, $b = 1$, $c = 0$, $x = u\sqrt{e_1 - e_3}$, $\bar{B} = [1/(e_1 - e_3)][B + e_3 n(n+1)]$; then we obtain

$$\frac{d^2R}{dx^2} - [n(n+1)k^2 \operatorname{sn}^2 x + \bar{B}]R = 0,$$

which is Jacobi's form.

PARTICULAR SOLUTIONS

We try to satisfy equation 64 by a polynominal R of degree n; the solutions belong to four classes:

$$\text{I: } R=f(\rho^2), \qquad n=2k$$

$$\text{II: } R=\sqrt{\rho^2-a^2}f(\rho^2), \quad R=\sqrt{\rho^2-b^2}f(\rho^2), \quad R=\sqrt{\rho^2-c^2}f(\rho^2), \qquad n=2k+1,$$

$$\text{III: } R=\sqrt{(\rho^2-b^2)(\rho^2-c^2)}f(\rho^2), \quad \ldots, \quad \ldots, \qquad n=2k+2,$$

$$\text{IV: } R=\sqrt{(\rho^2-a^2)(\rho^2-b^2)(\rho^2-c^2)}f(\rho^2), \qquad n=2k+3,$$

choosing constant B suitably. Otherwise the solution is a doubly periodic function of the second species by Hermite. Note: The above values of n for class III ($2k+2$) and class IV ($2k+3$) belong to what Appell (1921, p. 136) calls the "general picture (Tableau général)" for those classes. In the special discussion which follows (here, as in Appell), $n=2k$ for class III and $n=2k+1$ for class IV.

Note also that a given class may have more than one "form"; these are indicated: II_1, II_2, etc.

Class I:

$$R=f(\wp u)=\wp^k u+\alpha_1\wp^{k-1}u+\ldots+\alpha_{k-1}(\wp u)+\alpha_k, \qquad n=2k.$$

Substitute this expression for R in equation 64 and use equations 51 and 52 for $\wp'$ and $\wp''$; then it is evident from the coefficient of $\wp^{k+1}$ that $n=2k$. From the coefficient of $\wp^k$, we obtain

$$4(k-1)(k-2)\alpha_1+6(k-1)\alpha_1=n(n+1)\alpha_1+B, \qquad \text{or} \qquad B=-2(2n-1)\alpha_1. \tag{65}$$

From the coefficient of $\wp^0$ we obtain the characteristic equation, which is an algebraic equation of degree $k+1$ for class I:

$$C^{I}_{k+1}(B)=0. \tag{66}$$

This equation is obtained after substituting the values of $\alpha_1, \ldots, \alpha_k$ determined by the equations obtained from the coefficients of $\wp^{k-1}, \ldots, \wp^1$.

Class III:

$$R=\sqrt{\wp u-e_2}\,\sqrt{\wp u-e_3}\,(\wp^{k-1}u+\beta_1\wp^{k-2}u+\ldots+\beta_{k-1}), \qquad n=2k.$$

The characteristic equations are of degree k with regard to B for three forms of class III:

$$C^{\text{III}_1}_k(B)=0, \qquad C^{\text{III}_2}_k(B)=0, \qquad C^{\text{III}_3}_k(B)=0. \tag{67}$$

Thus, when n is even, the number of roots of equation 66 is $k+1$, and the number of roots of equation 67 is $3k$; in total, the number of roots B is $4k+1=2n+1$.

Class II:

$$R=\sqrt{\wp u-e_1}(\wp^k+\alpha_1\wp^{k+1}+\ldots+\alpha_k), \qquad n=2k+1.$$

The characteristic equations are of degree $k+1$:

$$C^{\mathrm{II}_1}_{k+1}(B)=0, \qquad C^{\mathrm{II}_2}_{k+1}(B)=0, \qquad C^{\mathrm{II}_3}_{k+1}(B)=0. \tag{68}$$

Class IV:

$$R=\sqrt{\wp u-e_1}\ \sqrt{\wp u-e_2}\ \sqrt{\wp u-e_3}\ (\wp^{k-1}+\ .\ .\ .\ +\beta_{k-1}), \qquad n=2k+1.$$

The characteristic equation is of degree k:

$$C^{\mathrm{IV}}_k(B)=0. \tag{69}$$

Thus, when n is odd, we have $3(k+1)$ solutions for B of the second class and k solutions of the fourth class; in total, we have $4k+3=2n+1$ solutions. Thus in either of the cases, we obtain $2n+1$ values of B; hence $2n+1$ *Lamé* functions

$$R^1_nR^2_n\ .\ .\ .\ R^i_n\ .\ .\ .\ R^{2n+1}_n. \tag{70}$$

Hence there are $2n+1$ products $R^i_nM^i_nN^i_n$, or $2n+1$ Lamé polynomials $Q^1_n(x, y, z)$, . . ., $Q^{2n+1}_n(x, y, z)$, each of which is a linear combination of the Lamé products

$$Q(x, y, z)=\sum_{i=1}^{2n+1} A_iR^i_nM^i_nN^i_n,$$

where A_i are $2n+1$ arbitrary constants, and satisfy $\Delta V=0$.

It can be shown that the roots of the characteristic equation are all real and distinct and that the $2n+1$ Lamé functions are linearly independent. We give the expressions for the Lamé functions of low degree. (The order follows the value of n rather than class number.)
$n=1$: $2n+1=3$ functions:

$$R=\sqrt{\wp u-e_1}, \qquad R=\sqrt{\wp u-e_2}, \qquad R=\sqrt{\wp u-e_3};$$

or

$$R=\sqrt{\rho^2-a^2}, \qquad R=\sqrt{\rho^2-b^2}, \qquad R=\sqrt{\rho^2-c^2},$$

$$RMN=\sqrt{\wp u-e_1}\ \sqrt{\wp v-e_1}\ \sqrt{\wp w-e_1}=Cx, \qquad C=\text{constant}.$$

$n=2$: $2n+1=5$ functions:

Class III:

$$R=\sqrt{\wp u-e_1}\ \sqrt{\wp u-e_2},\ .\ .\ .,\ .\ .\ .,$$

$$RMN=\sqrt{\wp u-e_1}\ \sqrt{\wp v-e_1}\ \sqrt{\wp w-e_1}\ \sqrt{\wp u-e_2}\ \sqrt{\wp v-e_2}\ \sqrt{\wp w-e_2}.$$

$$=Cxy.$$

Class I:

$$\frac{d^2R}{du^2}=(6\wp u+B)=R.$$

Put $R=\wp u+\alpha$, then $d^2R/du^2=\wp''u=6\wp^2-g_2/2$ and $6\wp^2-g_2/2=(6\wp+B)(\wp+\alpha)$. Then $6\alpha+B=0$ and $\alpha B=-g_2/2$. The characteristic equation is $B^2=3g_2$.

Hence we have two functions

$$R=\wp u \pm \frac{1}{6}\sqrt{3g_2}.$$

$n=3$: $2n+1=7$ functions:
Class IV:

$$\wp' u,$$

Class II:

$$\left(\wp u+\frac{e_\alpha}{2}-\frac{B}{10}\right)\sqrt{\wp u-e_\alpha} \qquad (\alpha=1, 2, 3).$$

The characteristic equation is $B^2-6Be_1+45e_1^2-15g_2=0$.
We have 2×3 functions of class II.
$n=4$: $2n+1=9$ functions:
Class I:

$$\wp'' u-\frac{2}{7}B\wp u+\frac{3}{140}B^2-\frac{2}{5}g_2.$$

The characteristic equation $B^3-52g_2B+560g_3=0$ has three roots; we have three functions of class I.
Class III:

$$\left(\wp u-\frac{e_\alpha}{2}-\frac{B}{14}\right)\sqrt{(\wp u-e_\beta)(\wp u-e_\gamma)}.$$

The characteristic equation $B^2+10Be_1-35e_1^2-7g_2=0$ has two roots; we have six functions of class III.

LAMÉ FUNCTIONS AND SPHERICAL FUNCTIONS

We derive various properties of Lamé functions by comparing them with spherical functions. Put

$$x=x_1\sqrt{\rho^2-a^2}, \qquad y=y_1\sqrt{\rho^2-b^2}, \qquad z=z_1\sqrt{\rho^2-c^2}; \tag{71}$$

then $x_1^2+y_1^2+z_1^2-1=0$. As μ, ν vary, (x, y, z) describes an ellipsoid, and (x_1, y_1, z_1) a sphere. We have

$$x_1^2=\frac{(\mu^2-a^2)(\nu^2-a^2)}{(b^2-a^2)(c^2-a^2)}, \qquad y_1^2=\frac{(\mu^2-b^2)(\nu^2-b^2)}{(c^2-b^2)(a^2-b^2)}, \qquad z_1^2=\frac{(\mu^2-c^2)(\nu^2-c^2)}{(a^2-c^2)(b^2-c^2)}. \tag{72}$$

Put

$$x_1=\sin\theta\cos\varphi, \qquad y_1=\sin\theta\sin\varphi, \qquad z_1=\cos\theta; \tag{73}$$

then the Lamé product MN of order n, being a function of μ and ν, can be expressed in terms of θ and φ in the form

$$MN=P_n(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)=Y_n. \tag{74}$$

In fact, let the corresponding polynomial for RMN be Q_n and separate it into a sum of homogeneous polynomials

$$Q_n(x, y, z) = P_n(x, y, z) + P_{n-1}(x, y, z) + \ldots + P_0.$$

Then

$$RMN = \rho^n P_n\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right) + \rho^{n-1} P_{n-1}\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right) + \ldots$$

$$= \rho^n P_n\left(\frac{\sqrt{\rho^2 - a^2}}{\rho} \sin\theta \cos\varphi, \quad \frac{\sqrt{\rho^2 - b^2}}{\rho} \sin\theta \sin\varphi, \quad \frac{\sqrt{\rho^2 - c^2}}{\rho} \cos\theta\right) + \ldots.$$

For $\rho \to \infty$ we have

$$\frac{R}{\rho^n} \to 1 \quad \text{and} \quad \frac{\sqrt{\rho^2 - a^2}}{\rho} \to 1, \quad \ldots, \quad \ldots,$$

and we obtain equation 74.

Conversely, the most general spherical functions are written

$$Y_n(\theta, \varphi) = \sum_{i=1}^{2n+1} A_i M_n^i N_n^i, \tag{75}$$

or, more generally,

$$\Phi(\mu, \nu) = \sum_{n=0}^{\infty} Y_n = \sum_{n=0}^{\infty} \sum_{i=1}^{2n+1} A_n^i M_n^i N_n^i. \tag{76}$$

Suppose that $a \to b$; take $b^2 = a^2 - \epsilon$, $\mu^2 = a^2 - \epsilon\mu'^2$, $0 < \mu' < 1$, $\epsilon \to 0$. Then

$$x = \sqrt{\rho^2 - a^2}\, \mu' \sqrt{\frac{a^2 - \nu^2}{a^2 - c^2}}, \qquad y = \sqrt{\rho^2 - a^2}\sqrt{1 - \mu'^2}\sqrt{\frac{a^2 - \nu^2}{a^2 - c^2}}, \qquad z = \sqrt{\rho^2 - c^2}\sqrt{\frac{\nu^2 - c^2}{a^2 - c^2}}.$$

Putting $\mu' = \cos\varphi$, $\sqrt{\rho^2 - a^2} = r_1$, $\sqrt{\rho^2 - c^2} = r_2$ we obtain

$$\sin\theta = \sqrt{\frac{a^2 - \nu^2}{a^2 - c^2}}, \qquad \cos\theta = \sqrt{\frac{\nu^2 - c^2}{a^2 - c^2}} \tag{77}$$

and

$$x = r_1 \sin\theta \cos\varphi, \qquad y = r_1 \sin\theta \sin\varphi, \qquad z = r_2 \cos\theta. \tag{78}$$

Let $a \to b$; then $y/x = \tan\varphi$; we arrive by this degeneration at a function of θ and φ. The product $MN = f(\mu)f(\nu)$ satisfies the equation for a spherical function, and reduces to $Y_n(\theta, \varphi)$. The spherical harmonics are

$$X_n^r \sin p\varphi, \qquad X_n^r \cos p\varphi, \qquad (p = 0, 1, \ldots, n), \tag{79}$$

where

$$X_n^p = c(1 - t^2)^{n/2} \frac{d^{n+p}(1 - t^2)^n}{dt^{n+p}}, \qquad t = \cos\theta = \sqrt{\frac{\nu^2 - c^2}{a^2 - c^2}}.$$

Thus

$$M \to \begin{matrix}\cos \\ \sin\end{matrix}\, p\varphi, \qquad N \to X_n^p.$$

Class I: $n=2k$ and $M\rightarrow f_1(\mu'^2)=f_1(\cos^2\varphi)=\cos p\varphi$, since n is even. Thus the limiting values of M are

$$\cos 0=1,\ \cos 2\varphi,\ \cos 4\varphi,\ \ldots,\qquad \cos 2k\varphi=\cos n\varphi; \tag{80}$$

and those of N are

$$X_n^0,\ X_n^2,\ \ldots,\ X_n^n. \tag{81}$$

The Lamé equation

$$\frac{d^2M}{dv^2}=[n(n+1)\wp v+B]M,$$

after substitution of $\mu^2=a^2-\epsilon\mu'^2$, $a^2-b^2=\epsilon$, $\mu'=\cos\varphi$, becomes $d^2M/d\varphi^2=\text{constant } M$, with

$$\text{constant}=-p^2=-\frac{n(n+1)a^2+K}{a^2-c^2}\qquad (p=0,2,4,\ldots,n),$$

$$B=n(n+1)h+K.$$

The value of constant K is determined by

$$\left.\begin{aligned}\frac{n(n+1)a^2+K_1}{a^2-c^2}&=0,\\ &\cdots\\ \frac{n(n+1)a^2+K_2}{a^2-c^2}&=2^2,\\ \frac{n(n+1)a^2+K_{k+1}}{a^2-c^2}&=(2k)^2\end{aligned}\right\} \tag{82}$$

Class III: $n=2k$, and $M\equiv\sqrt{\mu^2-b^2}\ \sqrt{\mu^2-c^2}f(\mu^2)\rightarrow\sqrt{\epsilon(1-\mu'^2)(a^2-c^2)}f_1(\mu'^2)$; f is of degree $k-1$, and $M\rightarrow\sin p\varphi$, $(p=1,3\ \ldots,\ 2k-1)$.

The characteristic equation is

$$\frac{n(n+1)a^2+K}{a^2-c^2}=p^2\qquad (p=1,3,\ldots,2k-1). \tag{83}$$

There are k roots for each class III_1, III_2, and III_3. Similarly, we obtain $2n+1$ equations of the form of equations 82 and 83 for n odd, 2nd class and 4th class.

Next, suppose that $b\rightarrow c$; we take $b^2=c^2+\epsilon$ and $\nu^2=c^2+\epsilon\nu'^2$. Everything proceeds similarly.

THEOREMS

All roots of the characteristic equations are real (Liouville, 1841). Suppose that n is even: then there are $k+1$ solutions of type I, and k solutions of type III_1. Let the coefficients of the highest degree terms of B be equal to 1. Let b vary continuously between a and c by fixing a and c. The coefficients, and hence the roots, vary continuously, and imaginary roots can appear at first when two consecutive real roots become equal. Let $b \to a$; when $b = a$, the characteristic equations reduce to the form of equations 82 and 83. The roots of equation 83 are bisected by the roots of equation 82:

$$B_1 B_2 \ . \ . \ . \ B_k B_{k+1}, \tag{83a}$$

$$B_1' B_2' \ . \ . \ . \ B_k'. \tag{83b}$$

Let b vary continuously from a to c (the roots vary continuously). Suppose that $B_1 = B_2 = B_1'$, since B_1' is always between B_1 and B_2; that is, a double root of equation 83a would then coincide with a simple root of equation 83b. On this assumption, we should have

$$\frac{d^2M}{dv^2} = [n(n+1)\wp v + B_1]M, \qquad \frac{d^2M'}{dv^2} = [n(n+1)\wp v + B_1']M',$$

with $B_1 = B_1'$. Then

$$M'\frac{d^2M}{dv^2} - M\frac{d^2M'}{dv^2} = 0, \qquad M'\frac{dM}{dv} - M\frac{dM'}{dv} = \text{constant}.$$

Here

$$M = f(\wp v), \qquad M' = \sqrt{\wp v - e_2}\,\sqrt{\wp v - e_3}\, f_1(\wp v) = \wp_2 v \wp_3 v f_1(\wp v).$$

If we add 2ω or $2\omega'$ to v, then M and dM/dv remain unaltered, but M' and dM'/dv change their signs. Hence $M'(dM/dv) - M(dM'/dv)$ would change its sign. Thus we should have $M'(dM/dv) - M(dM'/dv) = 0$, or $(M'/M) = \text{constant}$; this is impossible. Accordingly, equations 83a and 83b cannot have any equal root, and equation 83a cannot have any double roots. Therefore the roots are always real. If the axes of the ellipsoid are imaginary, then the roots may be double, and $2n+1$ functions of order n may then coincide (Cohn, 1888).

The $2n+1$ Lamé functions of order n are linearly independent. Suppose that there were a relation

$$\sum_{i=1}^{r} \gamma_i R_i = 0, \qquad r < 2n+1,$$

with constants γ_i. Carry out the operation

$$\frac{d^2}{du^2} - n(n+1)\wp u$$

and repeat it $(r+1)$ times. Then, since

$$\frac{d^2R}{du^2} - [n(n+1)\wp u + B]R = 0,$$

we obtain

$$\sum_{i=1}^{r} \gamma_i (B_i)^s R_i \equiv 0 \qquad (s = 1, 2, \ . \ . \ ., r-1),$$

where B_i is the corresponding value of B for the function R_i.

Eliminating $\gamma_i R_i (i=1, 2, \ldots, r)$ from these r equations, we obtain

$$\begin{vmatrix} 1 & 1 \ldots & 1 \\ B_1 & B_2 \ldots & B_r \\ \cdots & \cdots & \cdots \\ (B_1)^{r-1} & (B_2)^{r-1} \ldots & (B_r)^{r-1} \end{vmatrix} = 0 \equiv \prod_{\substack{i \neq j \\ i,j}}^{1 \ldots r}{}' (B_i - B_j).$$

As we have proved, there is no equal root. Hence this is contradictory, and these Lamé equations are linearly independent.

DEVELOPMENT IN LAMÉ FUNCTIONS

Write

$$\ell^2 = \frac{1}{(\rho^2-\mu^2)(\rho^2-\nu^2)}, \qquad \ell_0^2 = \frac{1}{(\rho_0^2-\mu^2)(\rho_0^2-\nu^2)}, \tag{84}$$

and suppose that the ellipsoid E_0 is for $\rho = \rho_0$. Then,

$$\iint_{E_0} \ell_0 MNM_1N_1 d\sigma = 0, \tag{85}$$

where the integration is extended over the surface E_0; MN and M_1N_1 are different Lamé products either of the same order or of different orders. This relation (equation 85) shows that the system of Lamé products is a set of orthogonal functions. In fact, $V = RMN$ and $V_1 = R_1M_1N_1$ are potentials and satisfy $\Delta V = \Delta V_1 = 0$. By Green's theorem,

$$\int_T (V\Delta V_1 - V_1 \Delta V) d\tau = \int_{E_0} \left(V \frac{\partial V_1}{\partial n} - V_1 \frac{\partial V}{\partial n} \right) d\sigma = 0.$$

Suppose that ρ increases in the outward normal; then $dn = \alpha d\rho$, and $V' = \partial V / \partial n$ by equation 47. Hence

$$\frac{\partial V}{\partial n} = \frac{\partial V}{\alpha \partial \rho} = \frac{1}{\alpha} \frac{\partial V}{\partial u} \frac{du}{d\rho} = \frac{1}{\alpha} \frac{V'}{A} = \frac{V'}{\sqrt{(\rho^2-\mu^2)(\rho^2-\nu^2)}} = \ell R' MN. \tag{86}$$

Similarly, $\partial V_1 / \partial n = \ell R_1' M_1 N_1$. Consequently, the Green formula becomes

$$\int_{E_0} \ell (RMNR_1'M_1N_1 - R_1M_1N_1R'MN) d\sigma = 0.$$

For $\rho = \rho_0$, the Lamé functions R, R', R_1, R_1' are constant. Hence the system of Lamé products is a set of orthogonal functions. Note that we have proved the relation

$$\frac{du}{dn} = -\ell. \tag{87}$$

From this theorem, we can develop any arbitrary function over the surface of the ellipsoid $\rho = \rho_0$. Suppose that

$$\Phi(\mu, \nu) = \sum_{0}^{\infty} A_k M_k N_k. \tag{88}$$

From equation 85, we have

$$\int_{E_0} \Phi(\mu, \nu) \ell_0 M_k N_k d\sigma = A_k \int_{E_0} \ell_0 M_k^2 N_k^2 d\sigma. \tag{89}$$

Now $ds_1 = \beta d\mu$, $ds_2 = \gamma d\nu$, $d\sigma = ds_1 ds_2 = \beta\gamma d\mu d\nu$, or

$$d\sigma = \frac{\mu^2 - \nu^2}{\ell_0 \sqrt{-B^2C^2}} d\mu d\nu.$$

From equation 89, we obtain

$$\int_{E_0} \Phi(\mu, \nu) M_k N_k \frac{\mu^2 - \nu^2}{\sqrt{-B^2C^2}} d\mu d\nu = A_k \int_{E_0} M_k^2 N_k^2 \frac{\mu^2 - \nu^2}{\sqrt{-B^2C^2}} d\mu d\nu. \tag{90}$$

Thus the coefficient A_k can be determined (Lindemann, 1882; Titchmarsh, 1946). Especially, if the degree of $\Phi(\mu, \nu)$ is lower than that of $M_k N_k$, then

$$\int_{E_0} \Phi(\mu\nu) \ell_0 M_k N_k d\sigma = 0, \qquad (k > n). \tag{91}$$

Furthermore, equation 85 can be written

$$\iint (\wp v - \wp w) M(v) N(w) M_1(v) N_1(w)\, dv dw = 0,$$

where the limits of the integration are $0 < (v - \omega)/i < 4\omega'/i 0 < w - \omega' < 4\omega$. Since there is no function other than any function of the Lamé products, such as M_1N_1, which satisfies equation 85, the system of Lamé products is *complete*, and equation 88 is *unique*. For the convergence of the series (equation 88), it is necessary and sufficient to have the convergence of the series

$$\sum_{k=1}^{\infty} A_k^2$$

in accordance with Riesz-Fischer's theorem. We can extend this to almost-everywhere convergence by Menchoff's theorem.

ZERO OF LAMÉ FUNCTIONS

A polynomial $f(\rho^2)$ of ρ^2 has a real distinct root between a^2 and c^2.

Theorem: Suppose that there were an imaginary root. Factorize $f(\rho^2)$ into a factor $\varphi(\rho^2)$ with real roots alone and a factor $\psi(\rho^2)$ with imaginary roots alone, so that $f(\rho^2) = \varphi(\rho^2)\psi(\rho^2)$; we know that $\psi(\rho^2)$ always has the same sign and $\varphi(\rho^2)$ changes its sign. Take $M_k = f(\mu^2)$, $N_k = f(\nu^2)$, and put $\Phi(\mu, \nu) = \varphi(\mu^2)\varphi(\nu^2)$:

$$\int_{E_0} \Phi(\mu\nu) \ell_0 M_k N_k d\sigma = \int_{E_0} [\varphi(\mu^2)\varphi(\nu^2)]^2 \psi(\mu^2)\psi(\nu^2)\, d\sigma.$$

But the degree of φ, and hence of Φ, is lower than that of M_k. Accordingly, this integral is zero by equation 91. But the right-hand side is positive. Hence it is impossible to have imaginary factors.

Next suppose that we have multiple roots. Factorize $f(\rho^2)$ into a factor $\varphi(\rho^2)$ with real roots alone and a factor $\varphi(\rho^2)$ with even multiple roots alone. We can carry out the proof similarly to the above, and find that it is impossible to have multiple roots.

Finally, factorize $f(\rho^2)$ into a factor $\varphi(\rho^2)$ with roots between a^2 and c^2 alone and a factor $\psi(\rho^2)$ with the roots outside of a^2 and c^2 alone. We can carry out the proof similarly and find that it is impossible to have roots outside a^2 and c^2. Note that the Legendre polynomials have roots between -1 and $+1$ and that the roots are real and distinct.

Stieltjes' theorem (Stieltjes, 1885): A Lamé function of degree n can be written

$$(\rho^2-a^2)^{\kappa_1}(\rho^2-b^2)^{\kappa_2}(\rho^2-c^2)^{\kappa_3}\prod_{p=1}^{m}(\rho^2-\alpha_p),$$

where κ_1, κ_2, κ_3 are 0 or 1/2, $\alpha_1, \ldots, \alpha_m$ are real and distinct and are distinct from either of a^2 or b^2 or c^2, and $n/2=m+\kappa_1+\kappa_2+\kappa_3$. The r-th function among these $m+1$ Lamé functions has its $r-1$ zeros between c^2 and b^2, and its remaining $m-r+1$ zeros between b^2 and a^2; thus the zeros $\alpha_1, \ldots, \alpha_m$ for these $m+1$ functions are contained between c^2 and a^2.

For proof, take real variables $\varphi_1, \ldots, \varphi_m$ such as

$$\begin{aligned} c^2 \leqslant \varphi_p \leqslant b^2 \qquad & (p=1, \ldots, r-1), \\ b^2 \leqslant \varphi_p \leqslant a^2 \qquad & (p=r, r+1, \ldots, m), \end{aligned}$$

and consider

$$\Pi=\prod_{p=1}^{m}\left(|\varphi_p-a^2|^{\kappa_1+1/4}\cdot|\varphi_p-b^2|^{\kappa_2+1/4}\cdot|\varphi_p-c^2|^{\kappa_3+1/4}\right)\cdot\prod_{p\neq q}|\varphi_p-\varphi_q|;$$

Π becomes 0 when all φ_p take their smallest values and when they take their largest values. If ϕ_p are all distinct from each other and are distinct from either a^2 or b^2 or c^2, then $\Pi > 0$; Π is continuously bounded and can reach its nonzero, positive upper bound. The maximum condition is

$$\frac{\partial \log \Pi}{\partial \varphi_1}=\frac{\partial \log \Pi}{\partial \varphi_2}=\ldots=0,$$

or

$$\frac{\kappa_1+1/4}{\varphi_p-a^2}+\frac{\kappa_2+1/4}{\varphi_p-b^2}+\frac{\kappa_3+1/4}{\varphi_p-c^2}+\sum_{q=1}^{m}{}'\frac{1}{\varphi_p-\varphi_q}=0, \qquad (p=1, 2, \ldots, m).$$

This is equation 60 by which we have determined $\alpha_1, \alpha_2, \ldots, \alpha_p$. Thus the equation for determining $\alpha_1, \ldots, \alpha_m$ is

$$\begin{aligned} c^2 < \alpha_p < b^2, \qquad & (p=1, 2, \ldots, r-1), \\ b^2 < \alpha_p < a^2, \qquad & (p=r, r+1, \ldots, m). \end{aligned}$$

Hence if r is one of 1, 2, . . ., $m+1$, then there is a Lamé function that has $r-1$ zeros between c^2 and b^2 and $m-r+1$ zeros between b^2 and a^2.

This theorem is a particular case of Klein's oscillation theorem (Klein, 1881); that is, in the equation

$$\frac{d}{dx}\left(K\frac{dy}{dx}\right)-Gy=0 \qquad \text{with} \qquad G=\ell(x)-(\lambda_0+\lambda_1 x+\ldots+\lambda_n x^n)g(x),$$

$g>0$, K, ℓ, g are continuous, and with the boundary conditions that

$$\alpha_r' y_r(a_r)-\alpha_r y_r'(a_r)=0, \qquad \beta_r' y_r(b_r)-\beta_r y_r'(b_r)=0, \qquad (r=0, 1, \ldots, n),$$

in the $n+1$ closed intervals $(a_0 b_0)$, . . ., $(a_n b_n)$, $a_0<b_0<a_1<b_1<\ldots<a_n<b_n$, we can determine uniquely the characteristic numbers $\lambda_0, \lambda_1, \ldots, \lambda_n$ so that the characteristic function y_r has just m_r zeros in $a_r<x<b_r$, where $m_0, m_1, \ldots, m_n$ are given positive numbers including zero. The value of λ for which the solution exists that is not identically zero is called the *characteristic value*, and the corresponding solution the *characteristic function.* Lamé products are characteristic functions, and the corresponding B are the characteristic values.

LAMÉ FUNCTION OF THE SECOND KIND

The Lamé functions R that we have derived are particular solutions of the Lamé equation; Hermite found other particular solutions S independent of R:

$$\frac{d^2S}{du^2}=[n(n+1)\wp u+B]S.$$

we obtain

$$R\frac{d^2S}{du^2}-S\frac{d^2R}{du^2}=0,$$

or

$$R\frac{dS}{du}-S\frac{dR}{du}=2n+1. \tag{92}$$

Such Lamé functions are *of the second kind.* Integrating this equation gives

$$S=R\int_0^u \frac{2n+1}{R^2}\,du. \tag{93}$$

The general solution of the Lamé equation is C_1R+C_2S, with arbitrary constants C_1 and C_2.

(1) SMN satisfies $\Delta V=0$ as does RMN:

$$S=R\int_\infty^\rho \frac{2n+1}{R^2}\,\frac{\rho d\rho}{\sqrt{(\rho^2-a^2)(\rho^2-b^2)(\rho^2-c^2)}}. \tag{94}$$

As ρ varies from ∞ to a, $S>0$ for all ρ. Since R is of degree n,

$$S=\rho^n\int_\infty^\rho\left(-\frac{2n+1}{\rho^{2n+2}}+\ldots\right)d\rho=\frac{1}{\rho^{n+1}}+\ldots.$$

While the development of ρ begins with ρ^n, S begins with $1/\rho^{n+1}$.

(2) Suppose that R_n is of the first class; then the poles of $1/[R(s)]^2$, where $s=\wp u$ from equation 49, are $u_1, \ldots, u_n$. Function $R(s)=\alpha_1(u-u_r)+\alpha_2(u-u_r)^2+\ldots$; but we see that $\alpha_2=0$ by inserting this series expansion in the differential equation. Hence the principal part of $1/[R(s)]^2$ is $1/\alpha_1^2(u-u_r)^2$ with residue 0. Thus we can determine A_r so that

$$[R(s)]^{-2}-\sum_{r=1}^{u} A_r\wp(u-u_r)$$

has no pole at the points congruent to any one of u_r. This doubly periodic expression without any pole is equal to a constant A by Liouville's theorem. Hence, by integrating we obtain

$$\int_0^u \frac{du}{[R(s)]^2}=Au-\sum_{r=1}^{n} A_r[\zeta(u-u_r)+\zeta(u_r)].$$

Since $R(s)=R(\wp u)$ is an even function of u, we can group the roots in pairs such that $u_{n-r}=-u_{r+1}$, and we have

$$\int_0^u \frac{du}{[R(s)]^2}=Au-\sum_{r=1}^{n/2} A_r[\zeta(u-u_r)+\zeta(u+u_r)]$$

$$=Au-2\zeta(u)\sum_{r=1}^{n/2} A_r-\sum_{r=1}^{n/2}\frac{A_r\wp'(u)}{\wp(u)-\wp(u_r)}.$$

Hence

$$S_n(s)=(2n+1)\left[Au-2\zeta(u)\sum_{r=1}^{n/2} A_r\right]R_n(s)+\wp'(u)w_{n/2-1}(s), \tag{95}$$

where $w_{n/2-1}(s)$ is a polynomial in s of degree $n/2-1$ (Liouville, 1845; Heine, 1845; Lindemann, 1882).

For computing the numerical values of Lamé functions, Scharma (1936) gave recurrence formulas for Lamé functions after the fashion of Whittaker (1929) for Mathieu functions and of Humbert (1917) for Legendre functions. Write the Lamé product by $E_n^t(u)E_n^t(v)E_n^t(w)$; then we obtain

$$\left.\begin{aligned}\alpha_{ns,\,mt}E_n^{s'}+\beta_{ns,\,mt}E_n^s&=\mu_{ns,\,mt}E_m^t,\\ \alpha_{mt,\,ns}E_m^{t'}+\beta_{mt,\,ns}E_m^t&=\mu_{mt,\,ns}E_n^s,\end{aligned}\right\} \tag{96}$$

where $\mu_{ns,\,mt}$ and $\mu_{mt,\,ns}$ are constant, such that $\alpha_{ns,\,mt}=\alpha_{mt,\,ns}$, and $\beta_{mt,\,ns}$ are determined by differential equations.

Whittaker (1914a, 1914b) obtained an integral equation for Lamé functions:

$$y(x)=\lambda\int_0^{4K}(\mathrm{dn}x\mathrm{dn}s+k\cosh\eta\mathrm{cn}x\mathrm{cn}s+kk'\sinh\eta\mathrm{sn}x\mathrm{sn}x)^n y(s)\,ds,$$

where η is a constant, $k^2=(a^2-b^2)/(a^2-c^2)$, $k^2+k'^2=1$. The method has been generalized to functions satisfying Sturm-Liouville's boundary value problems (Whittaker, 1914c; Ince, 1921, 1923).

NONALGEBRAIC SOLUTION

We have obtained the solution in the form of polynomials; we now seek the solution without such a restriction. For n=integer such a solution has been obtained by Hermite (1878) and Halphen (1888); the solution is called the *Heine-Hermite function.* For n equals a half-integer, the solution is called the *Heine-Wangerin function* (Brioschi, 1878; Halphen, 1888; and Crawford, 1895).

A uniform function which has no singularity other than poles is called *meromorphic.* If a meromorphic function is multiplied by a constant μ or μ' when the period 2ω or $2\omega'$ is added to the variable u, then the meromorphic function is said to be *doubly periodic of the second species:*

$$F(u+2\omega)=\mu F(u),\, F(u+2\omega')=\mu' F(u).$$

If $\Phi(u+2\omega)=e^{au+b}\Phi(u)$, $\Phi(u+2\omega')=e^{a'u+b'}\Phi(u)$ with constants a, b, a', b'; then F is called *doubly periodic of the third species* by Hermite. Such functions are written in the form

$$F(u)=Be^{\lambda u}\frac{H(u-b_1)\ \ .\ .\ .\ \ H(u-b_r)}{H(u-a_1)\ \ .\ .\ .\ \ H(u-a_r)},$$

$$\Phi(u)=Ae^{\alpha u^2+\beta u}\frac{H(u-b_1)\ \ .\ .\ .\ \ H(u-b_r)}{H(u-a_1)\ \ .\ .\ .\ \ H(u-a_r)},$$

where $\lambda=(1/2\omega)\log\mu$, and A, B, α, β are constants. Or by using Weierstrass σ function

$$\sigma(u)=\frac{H(u)}{H'(0)}\,e^{\eta u^2/(2\omega)},$$

we have

$$H(u)=B'e^{\lambda u}\frac{\sigma(u-b_1)\ \ .\ .\ .\ \ \sigma(u-b_r)}{\sigma(u-a_1)\ \ .\ .\ .\ \ \sigma(u-a_r)}.$$

Suppose that y is a doubly periodic function of the second species; $u=0$ is a pole of order n. If we can choose so that the roots of y are the same as the roots of y'', then y''/y is doubly periodic and has a pole at $u\equiv 0$. Hence it is of the form $n(n+1)\wp u+B$. Such a function y can then satisfy the Lamé equation. Take

$$y=\prod\frac{\sigma(u+a)}{\sigma(a)\sigma(u)}\,e^{-u\zeta a},$$

where Π extends over n factors:

$$\frac{y'}{y}=\sum\,[\zeta(u+a)-\zeta u-\zeta a]=\sum\,\frac{1}{2}\frac{\wp'u-\wp'a}{\wp u-\wp a},$$

or

$$\frac{y''}{y}=2n\wp u+\sum\,\wp a+\frac{1}{2}\sum\,\frac{(\wp'u-\wp'a)(\wp'u-\wp'b)}{(\wp u-\wp a)(\wp u-\wp b)}.$$

Decompose the last sum into simple elements; then

$$\frac{1}{2}\frac{(\wp'u-\wp'a)(\wp'u-\wp'b)}{(\wp u-\wp a)(\wp u-\wp b)}=2(\wp u+\wp a+\wp b)+\frac{\wp'a+\wp'b}{\wp a-\wp b}\,[\zeta(u+a)-\zeta(u+b)-\zeta a+\zeta b].$$

Collect the terms of the form $\zeta(u+a)-\zeta(a)$ and put the coefficient equal to zero. Similarly, operate on $\zeta(u+b)-\zeta(b)$. If we put

$$\wp a=\alpha,\qquad \wp'a=\alpha',\qquad \wp b=\beta,\qquad \wp'b=\beta',\ \ldots,$$

with

$$\alpha'^2=4\alpha^3-g_2\alpha-g_3,\qquad \beta'^2=4\beta^3-g_2\beta-g_3,\ \ldots,$$

then we obtain n equations

$$\begin{aligned}&\frac{\alpha'+\beta'}{\alpha-\beta}+\frac{\alpha'+\gamma'}{\alpha-\gamma}+\frac{\alpha'+\delta'}{\alpha-\delta}+\ldots=0,\\ &\frac{\beta'+\alpha'}{\beta-\alpha}+\frac{\beta'+\gamma'}{\beta-\gamma}+\ldots\qquad\quad=0,\\ &\ldots\qquad\qquad\qquad\qquad\qquad\quad;\end{aligned}$$

among these n equations, $n-1$ are independent. The remaining equation is $y''/y=n(n+1)\wp u+(2n-1)\Sigma\wp a$. Thus the nth equation is

$$(2n-1)(\alpha+\beta+\ldots)=B.$$

For $n=1$, we have

$$y=C\,\frac{\sigma(u-a)}{\sigma(u)}\,e^{-u\zeta a},\qquad \wp a=B,$$

and $y''=(2\wp u+B)y$.

For $n=2$,

$$y=C\,\frac{\sigma(u-a)\sigma(u+b)}{\sigma^2 u}\,e^{(\zeta a-\zeta b)u},$$

$$\wp'a+\wp'b=0,\qquad \wp a+\wp b=\tfrac{1}{3}B,$$

and

$$\alpha^2-\beta\alpha+\beta^2-\tfrac{1}{4}g_2=0.$$

These are Heine-Hermite functions (Halphen, 1888; Hermite, 1885).

For the Heine-Wangerin function, we put $(\nu^2-1/4)$ instead of $n(n+1)$ for the first type (Wangerin, 1904) and put $(2\nu+1)(2\nu+3)/4$ instead of $n(n+1)$ for the second type (Haentzschel, 1893):

$$\frac{d^2y}{du^2}=[H(\wp u-e_\lambda)-K]y,\qquad\text{(1st type)}$$

$$\frac{d^2y}{du^2}=\left[H\,\frac{(e_\lambda-e_\nu)(e_\lambda-e_\kappa)}{\wp u-e_\lambda}-K\right]y,\qquad\text{(2nd type).}$$

Compare Sparre (1883).

GENERALIZED LAMÉ FUNCTIONS

An arbitrary surface of the fourth order that has a circle as a double curve is called a *cyclid.* Darboux considered an orthogonal system from confocal cyclids (Darboux, 1887, 1910; Klein, 1893, 1926; Blaschke, 1929; Bôcher, 1894).

The equation of a three-dimensional sphere is

$$K(X^2+Y^2+Z^2)+2AX+2BY+2CZ+D=0,$$

where the radius is given by $\rho^2=(A^2+B^2+C^2-DK)/K^2$. This equation can be expressed in the form

$$2\alpha X+2\beta Y+2\gamma Z+\delta\frac{X^2+Y^2+Z^2-R^2}{R}+i\epsilon\frac{X^2+Y^2+Z^2+R^2}{R}=0,$$

where the coordinates of the center are

$$X_0=-\frac{\alpha R}{\delta+i\epsilon},\quad Y_0=-\frac{\beta R}{\delta+i\epsilon},\quad Z_0=-\frac{\gamma R}{\delta+i\epsilon},$$

and

$$\rho=\frac{R\sqrt{\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2}}{\delta+i\epsilon}.$$

If we take $\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2=1$, then the radius is $R/(\delta+i\epsilon)$. The condition for the orthogonality of two spheres is

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta'+\epsilon\epsilon'=0.$$

Consider five spheres, any two of which are orthogonal and let

$$\frac{S_k}{R_k}=2\alpha_kX+2\beta_kY+2\gamma_kZ+\delta_k\frac{X^2+Y^2+Z^2-R^2}{R}$$
$$+i\epsilon_k\frac{X^2+Y^2+Z^2+R^2}{R}=0\quad(k=1,2,\ldots,5),$$

where $\alpha_k^2+\beta_k^2+\gamma_k^2+\delta_k^2+\epsilon_k^2=1$, $\alpha_k\alpha_{k'}+\beta_k\beta_{k'}+\ldots+\epsilon_k\epsilon_{k'}=0$. We have

$$\sum_{k=1}^{5}\left(\frac{S_k}{R_k}\right)^2=\left(\frac{X^2+Y^2+Z^2-R^2}{R}\right)^2+\left(\frac{X^2+Y^2+Z^2+R^2}{iR}\right)^2+4X^2+4Y^2+4Z^2=0,$$

or

$$\sum_{k-1}^{5}\frac{S_k}{R_k^2}=-2.$$

Put $x_k=\lambda(S_k/R_k)$; then

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0.$$

Such x_k are called the *pentaspherical coordinates:*

$$ds^2 = \frac{\sum_{i=1}^{5} dx_i^2}{\left(\sum_{i=1}^{5} \frac{x_i}{R_i}\right)^2}.$$

A surface which is represented by

$$\sum_{i=1}^{5} \lambda_i x_i^2 = 0$$

is called a "cyclid" and is a surface of the fourth order in Cartesian coordinates. Similarly to the quadrics in Cartesian coordinates, the equation

$$\sum_{i=1}^{5} \frac{x_i^2}{\lambda - e_i} = 0$$

represents a triply orthogonal system of confocal cyclids, and a point in space is determined by ρ, μ, ν such that

$$\sum_{i=1}^{5} \frac{x_i^2}{\rho - e_i} = 0, \qquad \sum_{i=1}^{5} \frac{x_i^2}{\mu - e_i} = 0, \qquad \sum_{i=1}^{5} \frac{x_i^2}{\nu - e_i} = 0.$$

Let

$$V = \left(\frac{\sum \frac{x_i}{R_i}}{\sqrt{\sum e_i x_i^2}}\right)^{1/2} \Psi(\mu, \nu, \rho);$$

then $\Delta V = 0$ is satisfied if Ψ satisfies

$$(\rho - \nu) \frac{\partial^2 \Psi}{\partial u^2} + (\mu - \rho) \frac{\partial^2 \Psi}{\partial v^2} + (\nu - \mu) \frac{\partial^2 \Psi}{\partial w^2}$$
$$+ (\mu - \nu)(\nu - \rho)(\rho - \mu) \left[\frac{5}{4}(\mu + \nu + \rho) - \frac{3}{4} \sum e_i\right] \Psi = 0.$$

Or, letting $\Psi(\mu, \nu, \rho) = E'(\mu) E''(\nu) E'''(\rho)$, we obtain

$$\frac{d^2 E}{dt^2} = \left[-\frac{5}{4}\lambda^3 + \frac{3}{4}\left(\sum e_i\right)\lambda^2 + A\lambda + B\right] E,$$

where

$$t = \int \frac{d\lambda}{2\sqrt{f(\lambda)}}, \qquad f(\lambda) = (\lambda - e_1)(\lambda - e_2)(\lambda - e_3)(\lambda - e_4)(\lambda - e_5).$$

The generalized Lamé equation is

$$\frac{d^2y}{dx^2}+\frac{1}{2}\left(\frac{1}{x-e_1}+\ldots+\frac{1}{x-e_n}\right)\frac{dy}{dx}$$

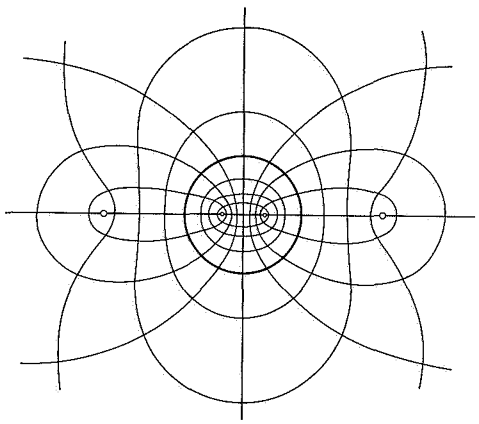
$$+\frac{1}{(x-e_1)\ \ldots\ (x-e_n)}\left[\frac{n(n-4)}{16}x^{n-2}+C_{n-3}x^{n-3}+\ldots+c_0\right]y=0;$$

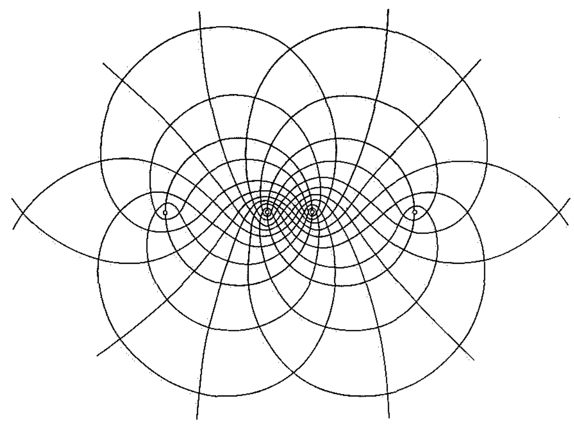
or

$$\frac{d^2y}{dt^2}=\left[-\frac{(n-4)}{4(n-1)}f''(x)+ax^{n-4}+bx^{n-5}+\ldots+m\right]y.$$

A study was made by Klein (1894, 1933) on the theory of monodromy groups, ikosahedron groups, etc. For Lamé functions see the works by Todhunter (1875); Heine (1878, 1881); Halphen (1888); Forsyth (1902); Poincaré (1885, 1902); Wangerin (1904); Appell (1921); Humbert (1926); Strutt (1932); McLachlin (1945).

These two figures are seen in Bôcher, 1894.





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CHAPTER IV

Theory of Poincaré

DIRICHLET PROBLEM ON AN ELLIPSOID

Let E_0 be an ellipsoid for $\rho=\rho_0$, and suppose that a harmonic function V_0 is given on the surface as a function of points on the surface:

$$V_0=\sum_0^\infty A_k M_k N_k=\sum_0^\infty \alpha_k R_k^0 S_k^0 M_k N_k, \tag{97}$$

$$A_k=\alpha_k R_k^0 S_k^0.$$

A harmonic function for the interior region is given by

$$V_i=\sum_0^\infty \alpha_k S_k^0 R_k M_k N_k, \tag{98}$$

and for the exterior region by

$$V_e=\sum_0^\infty \alpha_k R_k^0 S_k M_k N_k. \tag{99}$$

Since $R_k < R_k^0$ for $\rho < \rho_0$, and $S_k < S_k^0$ for $\rho > \rho_0$, the series in equations 98 and 99 are convergent if the series in equation 97 is convergent.

Suppose that the potentials V_0, V_i, V_e are caused by the surface density distribution ζ. Then

$$\frac{\partial V_e}{\partial n_e}+\frac{\partial V_i}{\partial n_i}=-4\pi\zeta. \tag{100}$$

Now

$$\frac{\partial V_e}{\partial n_e}=\frac{\partial V_e}{\partial u}\frac{du}{dn_e}=\sum \alpha_k R_k^0 \frac{dS_k}{du} M_k N_k \frac{du}{dn_e}=-\sum \alpha_k \ell_0 R_k^0 (S_k')_0 M_k N_k,$$

and

$$\frac{\partial V_i}{\partial n_i}=\sum \alpha_k S_k^0 \frac{dR_e}{du} M_k N_k \frac{du}{dn_i}=\sum \alpha_k \ell_0 S_k^0 (R_k')_0 M_k N_k.$$

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Then

$$\begin{aligned} -4\pi\zeta &= \sum \alpha_k \ell_0 M_k N_k (S_k R_k' - S_k' R_k)_0 \\ &= \ell_0 \sum_0^\infty (2n+1)\alpha_k M_k N_k, \end{aligned} \tag{101}$$

by Equation 92. Hence

$$\begin{aligned} V_0 &= \sum \alpha_k R_k^0 S_k^0 M_k N_k, \\ \zeta &= \ell_0 \sum \beta_k M_k N_k, \qquad \beta_k = \frac{2n+1}{4\pi}\alpha_k; \end{aligned} \tag{102}$$

we know ζ from α_k. Conversely when ζ is known, we obtain α_k from equation 102 and the potential from equations 97, 98, and 99.

As a special case,

$$\zeta = \epsilon \ell_0 M_k N_k, \qquad V_0 = \frac{4\pi}{2n+1}\epsilon R_k^0 S_k^0 M_k N_k,$$

or, since

$$V_0 = \int \frac{\zeta' d\sigma'}{\Delta} = \epsilon \int \frac{\ell_0' M_k' N_k' d\sigma'}{\Delta},$$

we have

$$\int \frac{\ell_0 M_k N_k d\sigma}{\Delta} = \frac{4\pi}{2n+1} R_k^0 S_k^0 M_k N_k \qquad \text{(Liouville).}$$

In particular, if $n=0$ and $R_0 = M_0 = N_0 = 1$, then

$$S = R\int_0^u \frac{2n+1}{R^2}\,du = \int_0^u du = u.$$

Hence the density distribution $\zeta = \epsilon\ell_0 M_0 N_0 = \epsilon\ell_0$ generates a potential $V_0 = 4\pi\epsilon u_0$, and, since $R_k = R_k^0 = 1$ and $S_k^0 = u_0$, $S_k = u$, we have

$$V_i = 4\pi\epsilon u_0, \qquad V_e = 4\pi\epsilon u. \tag{103}$$

The equipotential surface outside is $u=$ constant; i.e., $\rho = 0$ (i.e., confocal ellipsoids), and the potential inside is constant, which is in accord with Newton's theorem. Here $\zeta = \epsilon\ell_0$ denotes the thickness of the layer between two homothetic concentric ellipsoids, because

$$\frac{dC}{dn} = 2\sqrt{\frac{x^2}{(\rho_0 - a^2)^2} + \frac{y^2}{(\rho_0 - b^2)^2} + \frac{z^2}{(\rho_0 - c^2)^2}} = 2\frac{\alpha}{\rho} = \frac{2}{A_0\ell_0\rho_0} = \frac{K}{\rho_0}, \tag{104}$$

where

$$K = \frac{2}{A_0\rho_0}.$$

POTENTIAL OF AN ELLIPSOID

According to Poincaré, we put

for $n=0$:

$$R_0=1, \qquad M_0=1, \qquad N_0=1.$$

and for $n=1$: (105)

$$R_1=\sqrt{\rho^2-a^2}, \qquad R_2=\sqrt{\rho^2-b^2}, \qquad R_3=\sqrt{\rho^2-c^2}.$$

We have

$$x=h_1R_1M_1N_1, \qquad y=h_2R_2M_2N_2, \qquad z=h_3R_3M_3N_3,$$

where (106)

$$h_1^2=\frac{1}{(a^2-b^2)\,(a^2-c^2)}, \qquad h_2^2=\frac{1}{(b^2-a^2)\,(b^2-c^2)}, \qquad h_3^2=\frac{1}{(c^2-a^2)\,(c^2-b^2)}.$$

We put for $n=2$:

$$R_4=\sqrt{(\rho^2-b^2)\,(\rho^2-c^2)}, \qquad R_5=\sqrt{(\rho^2-a^2)\,(\rho^2-c^2)}, \qquad R_6=\sqrt{(\rho^2-a^2)\,(\rho^2-b^2)}, \tag{107}$$

$$R_7=\rho^2-\alpha_1, \qquad R_8=\rho^2-\alpha_2, \tag{108}$$

where $a^2>\alpha_1^2>b^2>\alpha_2^2>c^2$. However,

$$R_4=R_2R_3, \qquad R_5=R_3R_1, \qquad R_6=R_1R_2, \tag{109}$$

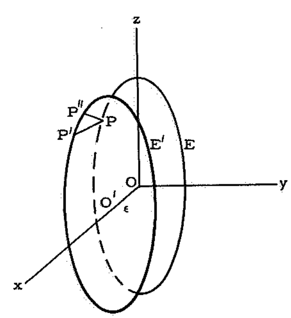
so that

$$yz=h_2h_3R_4M_4N_4, \qquad zx=h_3h_1R_5M_5N_5, \qquad xy=h_1h_2R_6M_6N_6, \tag{110}$$

and the volume of the ellipsoid is

$$T=\frac{4}{3}\,\pi R_1R_2R_3=\frac{4}{3}\,\pi R_1R_4=\frac{4}{3}\,\pi R_2R_5=\frac{4}{3}\,\pi R_3R_6; \tag{111}$$

$$\left.\begin{array}{l}\cos\,(n,x)=h_1\ell M_1N_1R_4, \qquad \cos\,(n,y)=h_2\ell M_2N_2R_5,\\ \qquad\qquad \cos\,(n,z)=h_3\ell M_3N_3R_6,\end{array}\right\} \tag{112}$$



where n is the external normal.

Now displace an ellipsoid E to an ellipsoid E' by a translation $\overline{OO'}=\epsilon$ along the x-axis. The potential V_1 due to E' at (x, y, z) is equal to the potential V due to E' at $(x-\epsilon, y, z)$.

Hence

$$V_1(x, y, z) = V(x-\epsilon, y, z) = V(x, y, z) - \epsilon \frac{\partial V}{\partial x}.$$

The difference at a point (x, y, z) of the potentials caused by E' and E is due to the potential $V' = V_1 - V = -\epsilon(\partial V/\partial x)$ of the surface layer distribution $\zeta = \overline{PP''}$. Now $\overline{PP'} = \epsilon$ and

$$\zeta = \epsilon \cos(\overline{PP'}, \overline{PP''}) = \epsilon \cos(n, x) = \epsilon h_1 R_4^0 \ell_0 M_1 N_1 \tag{113}$$

from equation 102. Hence, from equations 97 and 98, we obtain

$$\left.\begin{aligned} V_i' &= \frac{4\pi}{3} \epsilon h_1 R_4^0 S_1^0 R_1 M_1 N_1, \\ V_e' &= \frac{4\pi}{3} \epsilon h_1 R_4^0 R_1^0 S_1 M_1 N_1. \end{aligned}\right\} \tag{114}$$

From equation 106, we obtain

$$\frac{\partial V}{\partial x} = -\frac{1}{\epsilon} V_i' = -T \frac{S_1^0}{R_1^0} x, \qquad \frac{\partial V}{\partial y} = -T \frac{S_2^0}{R_2^0} y, \qquad \frac{\partial V}{\partial z} = -T \frac{S_3^0}{R_3^0} z. \tag{115}$$

Hence,

$$V_i = -\frac{T}{2} \left(\frac{S_1^0}{R_1^0} x^2 + \frac{S_2^0}{R_2^0} y^2 + \frac{S_3^0}{R_3^0} z^2 \right). \tag{116}$$

Similarly,

$$\frac{\partial V}{\partial x} = -\frac{1}{\epsilon} V_e' = -T \frac{S_1}{R_1} x, \qquad \frac{\partial V}{\partial y} = -T \frac{S_2}{R_2} y, \qquad \frac{\partial V}{\partial z} = -T \frac{S_3}{R_3} z. \tag{117}$$

ELLIPSOID AS AN EQUILIBRIUM FIGURE

Putting V in the equation of the equilibrium surface

$$U = V + \frac{\omega^2}{2}(x^2 + y^2) = \text{constant}, \tag{118}$$

we have

$$-T \frac{S_1^0}{R_1^0} x^2 + \left(\omega^2 - T \frac{S_2^0}{R_2^0}\right) y^2 + \left(\omega^2 - T \frac{S_3^0}{R_3^0}\right) z^2 = \text{constant},$$

which should coincide with

$$\frac{x^2}{(R_1^0)^2} + \frac{y^2}{(R_2^0)^2} + \frac{z^2}{(R_3^0)^2} = 1.$$

Thus the condition is

$$\frac{\omega^2}{T} = \frac{S_2 R_2 - S_1 R_1}{R_2^2} = \frac{S_3 R_3 - S_1 R_1}{R_3^2}; \tag{119}$$

or

$$\frac{\omega^2}{T} = \frac{S_2 R_2 - S_1 R_1}{R_2^2}, \tag{120}$$

and

$$\frac{S_2R_2-S_1R_1}{R_2^2}=\frac{S_3R_3-S_1R_1}{R_3^2}. \tag{121}$$

Note that

$$S_2R_2-S_1R_1>0, \qquad S_3R_3-S_1R_1>0. \tag{122}$$

For a Maclaurin spheroid $b=c$, equation 121 is satisfied. For a Jacobi ellipsoid, equation 121 can be put in the form

$$\frac{R_1S_1}{3}=\frac{R_4S_4}{5}. \tag{123}$$

CONDITION FOR AN EQUILIBRIUM FIGURE

Surface gravity $g=-dU/dn_e$ on the surface of an ellipsoid

$$U=K\left(\frac{x^2}{\rho_0^2-a^2}+\frac{y^2}{\rho_0^2-b^2}+\frac{z^2}{\rho_0^2-c^2}-1\right)+C$$

satisfies an important relation:

$$g\ell=\text{constant}. \tag{124}$$

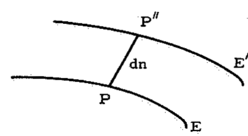
The value of the constant is determined at the pole, where $y=z=0$,

$$U=V, \qquad x=R_1^0, \qquad \mu=b, \qquad \nu=c, \qquad g=TS_1^0, \qquad \ell_0\ \frac{1}{R_0^2R_3^0}. \tag{125}$$

Hence,

$$g\ell_0=\frac{4}{3}\pi R_1^0S_1^0. \tag{126}$$

We deform E to another equilibrium figure E', which is represented by ζ. At a point P on E, we have



$$U_0=V_0+\frac{\omega^2}{2}(y_0^2+z_0^2)=\text{constant}.$$

At a point P'' on E', we have $U=U_0+(dU/dn)\zeta=U_0-g\zeta$. Let the potential due to the deformed layer be v; then the potential due to the deformed E' at P'' is

$$U+v=U_0-g\zeta+v=\text{constant}.$$

In order that E' be an equilibrium figure, we should have

$$v-g\zeta=\text{constant}. \tag{127}$$

Now we have by equation 102

$$\zeta=\sum_{k=0}^{\infty}\beta_k\ell_0M_kN_k,$$

where β_k and α_k are both constant and of degree 0. Hence, from equation 91 we have

$$\int \beta_k \ell_0 M_k N_k d\sigma = 0 \qquad (k=1, 2, \ldots).$$

Since the volume is constant,

$$\int \zeta d\sigma = 0,$$

or

$$\sum_{k=0}^{\infty} \int \beta_k \ell_0 M_k N_k d\sigma = 0,$$

or, by the above relation,

$$\int \beta_0 \ell_0 M_0 N_0 d\sigma = \beta_0 \int \ell_0 d\sigma = 0,$$

or $\beta_0 = 0$. Thus,

$$\zeta = \sum_{k=1}^{\infty}{}' \beta_k \ell_0 M_k N_k. \tag{128}$$

Furthermore,

$$v = \sum_{k=0}^{\infty} \alpha_k S_k^0 R_k^0 M_k N_k, \qquad \beta_k = \frac{2n+1}{4\pi} \alpha_k. \tag{129}$$

Inserting equations 126, 128, and 129 in equation 127, we obtain

$$\sum_{k=1}^{\infty} \beta_k \left(\frac{4\pi}{2n+1} R_k S_k - g\ell_0\right) M_k N_k = \sum_{k=1}^{\infty} 4\pi\beta_k \left(\frac{S_k R_k}{2n+1} - \frac{S_1 R_1}{3}\right) M_k N_k = \text{constant}.$$

This should hold for all values of μ and ν. Hence we should have

$$\beta_k \left(\frac{S_k R_k}{2n+1} - \frac{S_1 R_1}{3}\right) = 0 \qquad (k=1, 2, \ldots, \infty). \tag{130}$$

If the quantity F_k in the parentheses of equation 130 is not zero, then β_k must be zero. If all such quantities F_k are not zero, then all β_k should be zero, and there is no other equilibrium figure. We proceed to see whether $F_k = 0$. Where $n=1$: there are only R_1, R_2, R_3. For $k=1$, we have $\beta_1(S_1R_1 - S_1R_1) \equiv 0$. For $k=2$, 3, we have $\beta_2 = \beta_3 = 0$, since $S_2R_2 - S_1R_1 > 0$, $S_3R_3 - S_1R_1 > 0$ by equation 122. Where $n=2$; for $k=4$, we have

$$\beta_4 \left(\frac{S_4R_4}{5} - \frac{S_1R_1}{3}\right) = 0.$$

For a Jacobi ellipsoid, we have always $F_k = 0$ by equation 123; hence β_4 is arbitrary. But for $k=5, 6, 7, 8$, we should have $\beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$ since $F_k \neq 0$ in general. It can be shown that this β_4 does not produce any new equilibrium figure, since a figure with $\zeta = \beta_1 \ell_0 M_1 N_1$ is the same figure displaced by $\epsilon = \beta_1/(h_1 R_1^0)$ along the x-axis, and a figure with $\zeta = \beta_4 \ell_0 M_4 N_4$ is the same figure obtained by rotating through an angle $\delta\theta$ around the x-axis, where

$$\beta_4 = \frac{3Th_2h_3}{4\pi S_4^0}\,\delta\theta\left(\frac{S_3^0}{R_3^0} - \frac{S_2^0}{R_2^2}\right).$$

The question now arises whether we can have

$$\frac{R_kS_k}{2n+1} - \frac{R_1S_1}{3} = 0 \qquad (k=5, 6, \ldots).$$

If one of such relations holds, then there is an equilibrium figure in the vicinity of an ellipsoidal equilibrium figure, which is called the *ellipsoid of bifurcation.*

Poincaré called the expression

$$F = \frac{R_kS_k}{2n+1} - \frac{R_iS_i}{2n+1} = 0 \tag{131}$$

the *coefficient of stability* (Poincaré, 1885 and 1902). It can be shown that R_i/R_k should not always vary in the same sense in order to give an equilibrium figure. Since four among the eight forms of R_k are divisible by $R_1 = \sqrt{\rho^2 - a^2}$ and since R_k/R_1 keeps the same sign while ρ^2 varies from $+\infty$ to a^2, it can be shown also that there exists one and only one root of equation 131 with $m=1$ for the four forms without the factor $\sqrt{\rho^2 - a^2}$. For example, we take $n=2$ and $R_4 = \sqrt{(\rho^2 - b^2)(\rho^2 - c^2)}$. Now R_4 is not divisible by R_1. Hence

$$\frac{R_4S_4}{5} - \frac{R_1S_1}{3} = 0$$

has one and only one root of ρ^2. This is a Jacobi ellipsoid.

EQUILIBRIUM FIGURES DERIVED FROM MACLAURIN SPHEROIDS

Our conditions are now

$$R_2 = R_3, \qquad \frac{R_1S_1}{3} = \frac{R_kS_k}{2n+1}. \tag{132}$$

In this case, Lamé functions are reduced to spherical functions:

$$\left.\begin{aligned} R_k &= F(is) = h(1+s^2)^{p/2}\frac{d^{n+p}(1+s^2)^n}{ds^{n+p}}, \\ M_k &= F(t) = h(1-t^2)^{p/2}\frac{d^{n+p}(1-t^2)^n}{dt^{n+p}}, \qquad N_k = \frac{\cos}{\sin}p\phi, \end{aligned}\right\} \tag{133}$$

where

$$\frac{z}{y} = \tan\phi, \qquad t = \sqrt{\frac{a^2-\mu^2}{a^2-c^2}}, \qquad t = is.$$

In order that R_k not be divisible by R_1, that is, by s, we should have $n+p$ even. Take n to be even; then p should be even, and $0 < p < n$. For each value of p, we have two solutions

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$$M_k = F(t) = X_n^p, \qquad N_k = \cos p\varphi,$$

and

$$M_{k'} = X_n^p, \qquad N_{k'} = \sin p\varphi.$$

One of the two figures represented by these two solutions is obtained by rotating the other.

For $p=0$, we have

$$\frac{\zeta}{\ell_0} = \beta X_n^0, \qquad X_n^0 = h\frac{d^n(1-t^2)^n}{dt^n}, \qquad t = \sqrt{\frac{a^2-\mu^2}{a^2-c^2}}; \tag{134}$$

Quantity ζ becomes zero for n values of μ. The figure is symmetrical with respect to the yz-plane, and ζ vanishes and changes sign on the corresponding parallels. Neither $t=0$ nor $t=1$ is the root; this is a zonal figure.

For $p=n$, we have

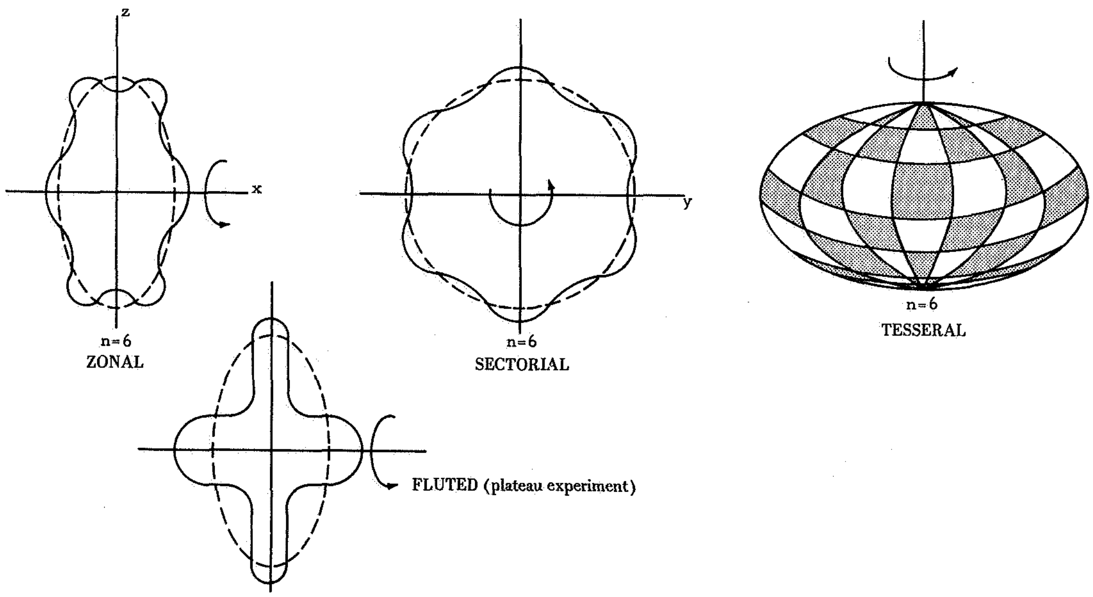
$$\frac{\zeta}{\ell_0} = \beta X_n^n \cos n\varphi, \qquad X_n^n = h_1(1-t^2)^{n/2}\frac{d^{2n}(1-t^2)^n}{dt^{2n}} = h_1(1-t^2)^{n/2}; \tag{135}$$

and ζ becomes zero on p meridian sections and on n-p parallels. This is a tesseral figure.

For a general value of p,

$$\frac{\zeta}{\ell_0} = \beta X_n^p \cos p\varphi, \tag{136}$$

and ζ becomes zero on p meridian sections and on n-p parallels. This is a tesseral figure.



We obtain ellipsoidal equilibrium figures in the vicinity of a Maclaurin spheroid by taking

$$R_4=\sqrt{(\rho^2-b^2)(\rho^2-c^2)}, \qquad R_7=\rho^2-\alpha_1, \qquad R_8=\rho^2-\alpha_2,$$

where α_1 and α_2 are the roots of

$$\frac{1}{\alpha-a^2}+\frac{1}{\alpha-b^2}+\frac{1}{\alpha-c^2}=0,$$

and $a^2 > \alpha_1 > b^2 > \alpha_2 > c^2$. For $R_k = R_4$, $n = 2$, $p = n$, we have $R_1S_1/3=R_4S_4/5$, $b=c$; this is both a Maclaurin spheroid and a Jacobi ellipsoid; hence it is the bifurcation figure. For $R_k=R_8$, we obtain the same figure rotated through $\pi/4$. For $R_k=R_7$ we have $n=2$, $p=0$; this is also a spheroid—the limiting Maclaurin spheroid.

Now

$$f^2=\frac{e^2}{1-e^2}=\frac{a^2-c^2}{\rho^2-a^2}=\frac{1}{s^2}.$$

For $f\to 0$, we have $\rho\to\infty$, $s\to\infty$; this is a sphere. As $f>0$ increases, the ellipticity of the meridian section increases. The first figure we meet for bifurcation can be shown to be the bifurcation figure with Jacobi ellipsoid. In fact, we can show that

$$\left.\begin{aligned} &\frac{R_1S_1}{3}>\frac{R_pS_p}{2n+1}>\frac{R_{p'}S_{p'}}{2n+1} \qquad \text{for } p'<p. \\ \text{and} \qquad &\\ &\frac{R_pS_p}{2n+1}>\frac{R_kS_k}{2n'+1} \qquad \text{for } n<n'. \end{aligned}\right\} \tag{137}$$

Thus $R_iS_i/(2n+1)$ decreases as the ellipticity of the meridian sections increases (Véronnet, 1920).

EQUILIBRIUM FIGURES DERIVED FROM JACOBI ELLIPSOIDS

Such figures must satisfy the two conditions

$$\frac{R_1S_1}{3}=\frac{R_kS_k}{2n+1}, \qquad \frac{R_4S_4}{5}=\frac{R_kS_k}{2n+1}, \tag{138}$$

and $n>2$, $k>8$. To satisfy these relations, neither R_k/R_1 nor R_k/R_4 should always vary in the same sense. Consider $n=$ even and take the first class. Then $R_k=f(\rho^2)$ and all roots of the polynomial $f(\rho^2)$ are real and contained between a^2 and c^2. Let α be the largest root, and $f(\rho^2)=(\rho^2-\alpha)(\rho^2-\alpha_1)$ . . .. Since $R_4=\sqrt{(\rho^2-b^2)(\rho^2-c^2)}$, we have

$$\left(\frac{R_k}{R_4}\right)^2=\frac{\rho^2-\alpha}{\rho^2-b^2}\frac{\rho^2-\alpha}{\rho^2-c^2}(\rho^2-\alpha_1)^2 \ . \ . \ ., \qquad \infty>\rho^2>a^2>b^2>c^2.$$

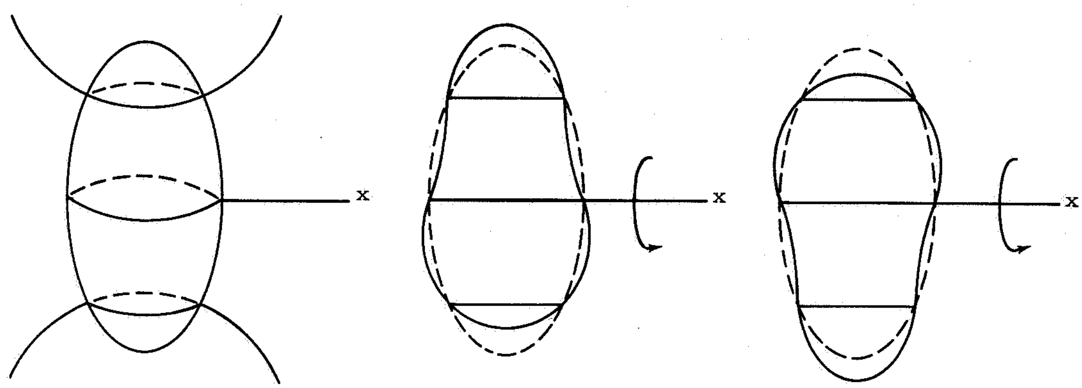
To satisfy our conditions, all roots α should be contained between b^2 and c^2. By Klein-Stieltjes' theorem, there is one and only one polynomial among $k+1$ polynomials of order $n=2k$ such that all its roots are contained between b^2 and c^2. Hence there is one and only one ellipsoid of bifurcation for any value of n; that is, of class I for n even and of class II of the form $R=\sqrt{\rho^2-c^2}[f(\rho^2)]$ for n odd.

For $n=2$, we have found the bifurcation ellipsoid at the junction with a Maclaurin Spheroid.

For $n=3$ we have $R_k=\sqrt{\rho^2-c^2}\,(\rho^2-\alpha)$, $b>\alpha>c^2$. We have

$$\frac{\zeta}{\ell_0}=\epsilon' z\left(\frac{x^2}{\alpha-a^2}+\frac{y^2}{\alpha-b^2}+\frac{z^2}{\alpha-c^2}-1\right).$$

This is a pear-shaped figure (Poincaré, 1885a, 1902a; Liapounov, 1884).



Darwin (1902) and Humbert (1918b) made the numerical computation. Let the three axes of the bifurcation Jacobi ellipsoid be A, B, and C, and put $ABC=1$; then $\alpha=0.57453$, $A/C=0.3451$, $B/C=0.4323$.

For $n=4$, we have

$$R_9=f(\rho^2)=(\rho^2-\alpha_1)(\rho^2-\alpha_2), \qquad b^2>\alpha_1>\alpha_2>c^2,$$

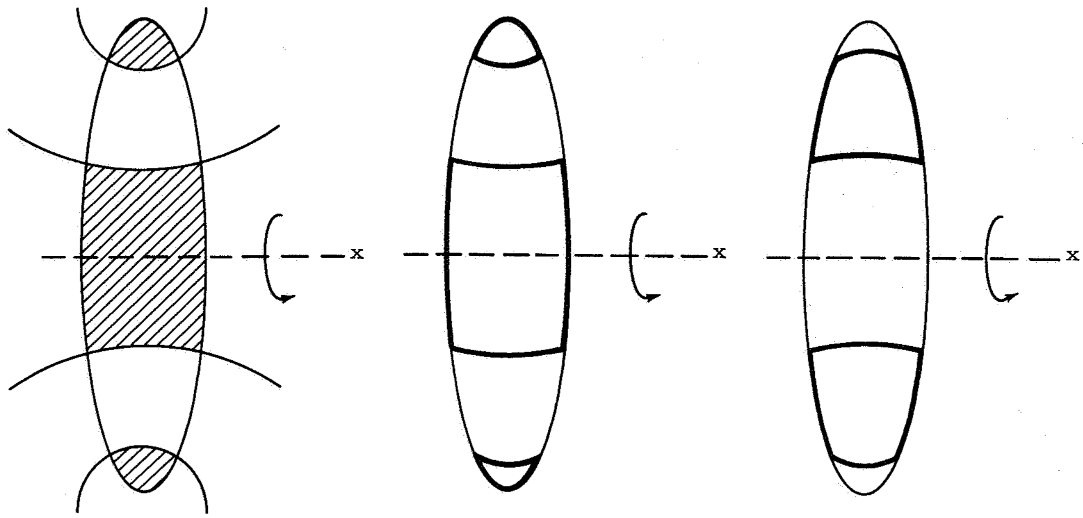
and

$$\zeta=\epsilon(\rho^2-\alpha_1)(\mu^2-\alpha_1)(\nu^2-\alpha_1)(\rho^2-\alpha_2)(\mu^2-\alpha_2)(\nu^2-\alpha_2)$$

$$=\epsilon'\left(\frac{x^2}{\alpha_1-a^2}+\frac{y^2}{\alpha_1-b^2}+\frac{z^2}{\alpha_1-c^2}-1\right)\left(\frac{x^2}{\alpha_2-a^2}+\frac{y^2}{\alpha_2-b^2}+\frac{z^2}{\alpha_2-c^2}-1\right);$$

$$A/C=0.2575, \qquad B/C=0.2970, \qquad \alpha_1=0.1142, \qquad \text{and } \alpha_2=0.7324.$$

The figures for $n=3$, 4 were computed by Liapounov and Humbert. The figures for $n=5$, 6 were computed by Humbert (1915, 1915a, 1916, 1917, 1918, 1918a, 1918b, 1919), referring to new recurrence formulas for computing Lamé functions of higher order.



For $n=5$ we take $R=\sqrt{(\rho^2-c^2)}\ (\rho^2-\alpha_1)(\rho^2-\alpha_2)$, with $\alpha_1=0.4539$, $\alpha_2=0.6545$. We obtain $A/C=0.1678$, $B/C=0.1810$.

For $n=6$ we take $R=(\rho^2-\alpha_1)(\rho^2-\alpha_2)(\rho^2-\alpha_3)$ with $\alpha_1=0.080$, $\alpha_2=0.423$, $\alpha_3=0.869$. We obtain $A/C=0.140$, $B/C=0.148$.

We thus see the ellipsoid of bifurcation becomes more and more elongated as we proceed to a higher value of n. Poincaré, from his analytic expression in the form of the series of Lamé functions, considered the pear-shaped figure to be stable, but Liapounov working independently proved it to be unstable. The complete works of Liapounov are described in chapter VI.

Appel (1910, 1913, and 1919) transformed the problem to the solution of an integral equation of Fredholm's type and proved the existence of the solution. Rotating figures due to surface tension were discussed by Globa-Mikhaïlenko (1916, 1919) and Charrueau (1926).

DARWIN'S ANALYSIS

Darwin (1901, 1902, 1903, 1910) denoted the three roots of

$$\frac{x^2}{a^2+u^2}+\frac{y^2}{b^2+u^2}+\frac{z^2}{c^2+u^2}=1$$

by

$$u_1^2=k^2\nu^2, \qquad u_2^2=k^2\mu^2, \qquad u_3^2=k^2\frac{1-\beta\cos 2\varphi}{1-\beta},$$

where

$$\infty\geqslant\nu\geqslant 0, \qquad +1\geqslant\mu\geqslant -1, \qquad 2\pi\geqslant\varphi\geqslant 0,$$

$$\infty\geqslant u_1^2>-a^2>u_3^2>-b^2>u_2^2>-c^2,$$

and

$$a^2=-k^2\frac{1+\beta}{1-\beta}, \qquad b^2=-k^2, \qquad c^2=0.$$

The Cartesian coordinates are written

$$\frac{x^2}{k^2}=-\frac{1-\beta}{1+\beta}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)\left(\mu^2-\frac{1+\beta}{1-\beta}\right)\cos^2\varphi,$$

$$\frac{y^2}{k^2}=-(\nu^2-1)(\mu^2-1)\sin^2\varphi,$$

$$\frac{z^2}{k^2}=\nu^2\mu^2\frac{1-\beta\cos 2\varphi}{1+\beta}.$$

Corresponding to spherical functions P_i^s, Q_i^s Darwin wrote $\mathfrak{P}_i^s$, $\mathfrak{Q}_i^s$, for both μ and ν, and corresponding to spherical functions $\frac{\sin}{\cos}s\varphi$ he put $\mathfrak{C}_i^s$, $\mathfrak{S}_i^s$. When k is imaginary, they become respectively P_i^s, Q_i^s, C_i^s, S_i^s. Table I shows eight classes.

TABLE I.—*Eight Classes as Distinguished by Darwin*

Class			i	s	Cosine or sine
O	or	EEC	even	even	cos
AB	or	EES	even	even	sin
A		OOC	odd	odd	cos
B		OOS	odd	odd	cos
C		OEC	odd	even	cos
ABC		OES	odd	even	sin
CA		EOC	even	odd	cos
CB		EOS	even	odd	sin

Let $\nu=\nu_0$ be the surface of the ellipsoid; then the potential is expressed as

$$V_i=\frac{4\pi k^2\rho\nu_0}{\mathfrak{C}_i^s}(\nu_0^2-1)^{1/2}\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)^{1/2}\mathfrak{P}_i^s(\nu)\mathfrak{Q}_i^s(\nu_0)\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\varphi).$$

$$V_e=\frac{4\pi k^2\rho\nu_0}{\mathfrak{C}_i^s}(\nu_0^2-1)^{1/2}\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)^{1/2}\mathfrak{P}_i^s(\nu_0)\mathfrak{Q}_i^s(\nu)\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\varphi).$$

The distorted ellipsoid is written

$$\frac{x^2}{k^2\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)}+\frac{y^2}{k^2(\nu_0^2-1)}+\frac{z^2}{k^2\nu_0^2}=1+2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\varphi).$$

The stability coefficient of Poincaré is written

$$\mathfrak{R}_i^s \equiv 1 - \frac{\mathfrak{P}_i^s(\nu_0)\mathfrak{Q}_i^s(\nu_0)}{P_1'(\nu_0)Q_1'(\nu_0)} = 0.$$

By such laborious computation, Darwin concluded after Poincaré that the pear-shaped figure was stable. But in his development, he missed a term in his expansion which exceeded numerically the preceding term. Actually the pear-shaped figure is unstable, as was proved by Liapounov.

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CHAPTER V

Theory of Jeans

METHOD OF JEANS

Jeans at first considered the stability of equilibrium figures of a cylinder rotating around its axis (Jeans, 1902). He proved that the two-dimensional analog of the pear-shaped figure is unstable, in opposition to Poincaré's conclusion for three-dimensional figures. He then studied three-dimensional figures without heeding the convergence of the expression (Jeans, 1915, 1916, 1919).

Take the reference ellipsoid as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and consider

$$f(x, y, z; \lambda) \equiv \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1=0. \tag{139}$$

The potential at a point (x, y, z) because of this solid ellipsoid is expressed by

$$V_e=\int_\lambda^\infty \psi(\lambda)f(x, y, z; \lambda)d\lambda, \qquad V_i=\int_0^\infty \psi(\lambda)f(x, y, z; \lambda)d\lambda, \tag{140}$$

where λ is the root of $f(x, y, z; \lambda)=0$ and

$$\psi(\lambda)=-\frac{\pi abc\rho}{\Delta}, \qquad \Delta=\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}. \tag{141}$$

Suppose that $f(x, y, z; \lambda)=0$ is distorted to $F(x, y, z; \lambda)=0$. Then under what condition can the potential be represented by the form

$$V_e=\int_\lambda^\infty \psi(\lambda)F(x, y, z; \lambda)d\lambda, \qquad V_i=\int_0^\infty \psi(\lambda)F(x, y, z; \lambda)d\lambda, \tag{142}$$

where the lower limit of the integral λ is the root of $F(x, y, z; \lambda)=0$? At all points in space F should satisfy $\nabla^2 V_e=0$, $\nabla^2 V_i=-4\pi\rho$, or

$$\nabla^2 V_e=\int_\lambda^\infty \psi(\lambda)\nabla^2 F d\lambda-\psi(\lambda)\sum\frac{\partial F}{\partial x}\frac{\partial \lambda}{\partial x},$$

$$\nabla^2 V_i=\int_0^\infty \psi(\lambda)\nabla^2 F d\lambda.$$

If F satisfies

$$\int_0^\lambda \psi(\lambda)\nabla^2 F d\lambda+\psi(\lambda)\sum\frac{\partial F}{\partial x}\frac{\partial \lambda}{\partial x}=-4\pi\rho \tag{143}$$

at all points in space, and if at $\lambda=\infty$ we have

$$\psi(\lambda)\sum\frac{\partial F}{\partial x}\frac{\partial \lambda}{\partial x}=0, \tag{144}$$

then equation 143 reduces at $\lambda=\infty$ to

$$\int_0^\infty \psi(\lambda)\nabla^2 F d\lambda=-4\pi\rho. \tag{145}$$

If F satisfies equations 143 and 144, then equation 145 is also satisfied at all points in space. Subtracting, we obtain

$$\int_\lambda^\infty \psi(\lambda)\nabla^2 F d\lambda-\psi(\lambda)\sum\frac{\partial F}{\partial x}\frac{\partial \lambda}{\partial x}=0. \tag{146}$$

Hence $\nabla^2 V_e=0$, $\nabla^2 V_i=-4\pi\rho$ are satisfied; λ is determined by $F=0$. With

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial \lambda}\frac{\partial \lambda}{\partial x}=0,$$

equation 143 is written in alternative form

$$\int_0^\lambda \psi(\lambda)\nabla^2 F d\lambda-\psi(\lambda)\frac{\sum\left(\frac{\partial F}{\partial x}\right)^2}{\frac{\partial F}{\partial \lambda}}=-4\pi\rho. \tag{147}$$

A solution of this equation such that the second terms vanishes at infinity defines the boundary of a solid whose potential is written in the form of equation 142. Certainly an ellipsoid $F=f$ satisfies this condition:

$$\int_0^\lambda \psi(\lambda)\nabla^2 f d\lambda-\psi(\lambda)\frac{\sum\left(\frac{\partial f}{\partial x}\right)^2}{\frac{\partial f}{\partial \lambda}}=-4\pi\rho,$$

is identically satisfied by the relation

$$\sum\left(\frac{\partial f}{\partial x}\right)^2=-4\frac{\partial f}{\partial \lambda}. \tag{148}$$

To obtain a more general solution, we put $F=f+\phi$; then the condition for ϕ, with $A=a^2+\lambda$, etc., becomes

$$\frac{\partial}{\partial \lambda}(f+\phi)\int_0^\lambda \psi(\lambda)\nabla^2\phi d\lambda=\psi(\lambda)\left[4\left(\sum\frac{x}{A}\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial \lambda}\right)+\sum\left(\frac{\partial \phi}{\partial x}\right)^2\right]. \tag{149}$$

Our problem is to solve this equation. Jeans put $\phi=u+fv$ and obtained the following equations for u and v:

$$\int_0^\lambda \psi(\lambda)\nabla^2(u+fv)d\lambda+4\psi(\lambda)v=0, \tag{150}$$

$$4(1+v)\left[\left(\sum\frac{x}{A}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial \lambda}\right)+f\left(\sum\frac{x}{A}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial \lambda}\right)\right]+\sum\left(\frac{\partial u}{\partial x}+f\frac{\partial v}{\partial x}\right)^2=0. \tag{151}$$

Consider equation 150 first. Note that we must have $v=0$ when $\lambda=0$. Take the boundary of the distorted ellipsoid to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+\phi_{\lambda=0}=0; \tag{152}$$

then we should have $\phi_{\lambda=0}=u_{\lambda=0}$ since $v=0$ for $\lambda=0$. From substituting equation 141 for $\psi(\lambda)$, we obtain

$$\int_0^{\lambda'}\left[\nabla^2u+f\nabla^2v+4\left(\sum\frac{x}{A}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial \lambda}\right)\right]\frac{d\lambda}{\Delta}=0, \tag{153}$$

where λ' is the value of λ satisfying $f+\phi=0$. The most general way of solving this equation is to put

$$\nabla^2u+f\nabla^2v+4\left(\sum\frac{x}{A}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial \lambda}\right)=\Delta\frac{\partial \sigma}{\partial \lambda}, \tag{154}$$

where σ is any function of x, y, z, λ, that vanishes for $\lambda=\lambda'$ and also for $\lambda=0$. Expand v in powers of f:

$$v=w+fw'+f^2w''+\ldots+f^nw^{(n)}+\ldots; \tag{155}$$

substitute on the left-hand side of equation 154; and equate the coefficients of successive powers of f.

$$\left.\begin{aligned}
&4\left(\sum\frac{x}{A}\frac{\partial w}{\partial x}+\frac{\partial w}{\partial \lambda}\right)=-\nabla^2u-\Delta\frac{\partial \theta}{\partial \lambda},\\
&4\left(\sum\frac{x}{A}\frac{\partial w'}{\partial x}+\frac{\partial w'}{\partial \lambda}\right)=-\frac{1}{2}\nabla^2w,\\
&\ldots\\
&4\left(\sum\frac{x}{A}\frac{\partial w^{(n)}}{\partial x}+\frac{\partial w^{(n)}}{\partial \lambda}\right)=-\frac{1}{n+1}\nabla^2w^{(n-1)}.
\end{aligned}\right\} \tag{156}$$

The indeterminate function θ should be chosen so that

$$\theta=\frac{4}{\Delta}\left(\frac{u}{1+v}\right)[w'+2fw''+\ .\ .\ .\ +(n+1)f^n w^{(n+1)}+\ .\ .\ .]. \tag{157}$$

Thus we obtain the solution v of equation 155 by solving equation 156 so that $w=w'=w''\ .\ .\ .\ =w^{(n)}=\ .\ .\ .\ =0$ for $\lambda=0$, with θ given by equation 157.

Next we turn to equation 151 and write

$$\xi=\frac{x}{A},\qquad \eta=\frac{y}{B},\qquad \zeta=\frac{z}{C},\qquad \lambda=\lambda; \tag{158}$$

then in the new coordinates with $f=-u/(1+v)$, we obtain

$$4\frac{\partial}{\partial\lambda}\left(\frac{u}{1+v}\right)+\sum\frac{1}{A^2}\left[\frac{\xi}{\partial\xi}\left(\frac{u}{1+v}\right)^2\right]=0. \tag{159}$$

This equation gives $u/(1+v)$. We know v; hence we obtain u and the unknown quantity ϕ and hence F.

Now Jeans expanded $u/(1+v)$ in powers of a parameter e:

$$\frac{u}{1+v}=eg_1+e^2g_2+\ .\ .\ .\ .$$

Substituting this expansion in equation 159 and equating successive powers of e give

$$\frac{\partial g_1}{\partial\lambda}=0,\qquad \frac{\partial g_2}{\partial\lambda}=-\frac{1}{4}\sum\frac{1}{A^2}\left(\frac{\partial g_1}{\partial\xi}\right)^2,\qquad \frac{\partial g_3}{\partial\lambda}=-\frac{1}{4}\sum\frac{1}{A^2}\left(2\frac{\partial g_1}{\partial\xi}\frac{\partial g_2}{\partial\xi}\right),\ .\ .\ .\ .$$

Put $g_1=P(\xi,\eta,\zeta)$, and write $\partial P/\partial\xi=P_\xi$, etc., and $\mathsf{A}=1/a^2-1/A$, etc. Then,

$$g_2=-(1/4)(\mathsf{A}P_\xi^2+\mathsf{B}P_\eta^2+\mathsf{C}P_\zeta^2)+Q(\xi,\eta,\zeta),$$

$$g_3=(1/8)(A^2P_\xi^2P_{\xi\xi}+\ .\ .\ .+2BCP_\eta P_\zeta P_{\eta\zeta}+\ .\ .\ .\ )-(1/2)(A\,P_\xi Q_\xi+\ .\ .\ .\ )+R(\xi,\eta,\zeta),$$

$.\ .\ .\ .$

Also, put $u=eu_1+e^2u_2+\ .\ .\ .\ ,v=ev_1+e^2v_2+\ .\ .\ .,\ w=ew_1+e^2w_2+\ .\ .\ .$; then we see that $u_1=g_1=P$, $u_2=g_2+v_1g_1$, $u_3=g_3+v_1g_2+v_2g_1$, $.\ .\ .$; $w_1=-DP/4$, $w_2=D^2P^2/64-DQ/4$, $.\ .\ .$; and

$$w_1^{(n)}=\frac{(-1)^{n+1}}{[(n+1)!]^2 4^{n+1}}=D^{n+1}P,$$

where

$$D=\mathsf{A}\frac{\partial^2}{\partial\xi^2}+\mathsf{B}\frac{\partial^2}{\partial\eta^2}+\mathsf{C}\frac{\partial^2}{\partial\zeta^2}.$$

Write $\phi = e\phi_1 + e^2\phi_2 + \ldots$; thus,

$$\phi_1 = u_1 + fw_1 + f^2 w_1' + \ldots$$
$$= P - (f/4)DP + \frac{1}{2^2}(f/4)^2 D^2 P - \frac{1}{(2^2)(3^2)}(f/4)^3 D^3 P + \ldots,$$
$$\phi_2 = u_2 + fv_2$$
$$= Q - \frac{1}{8}DP^2 + f\left(\frac{1}{64}D^2P^2 - \frac{1}{4}DQ\right) + f^2\left(-\frac{1}{1536}D^3P^2 + \frac{1}{64}D^2Q\right),$$
$$\phi = \phi_1 + \phi_2 + \ldots . \tag{160}$$

On the boundary $\lambda = 0$, we have

$$\phi_0 = eP_0 + e^2 Q_0 + \ldots . \tag{161}$$

This value of ϕ_0 can represent a general distortion of the fundamental ellipsoid. Jeans took a distorted ellipsoid to be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 + eP_0 + e^2 Q_0 + \ldots = 0 \tag{162}$$

and the potential to be

$$\left.\begin{aligned} V_i &= -\pi\rho abc \int_0^\infty \frac{f + e\phi_1}{\Delta} d\lambda, \\ \phi_1 &= P - \tfrac{1}{4}fDP + \tfrac{1}{4}(\tfrac{1}{4}f)^2 D^2 P \ldots . \end{aligned}\right\} \tag{163}$$

Put

$$\int_0^\infty \frac{d\lambda}{\Delta} = J, \qquad \int_0^\infty \frac{d\lambda}{A^m B^n C^p \Delta} = J_{A^m B^n C^p}.$$

In order that equation 162 be an equilibrium figure, $V_i + \omega^2(x^2+y^2)/2$ should be the free surface.

For $e=0$, we have

$$J_A - \frac{\omega^2}{2\pi\rho abc} = \frac{\theta}{a^2}, \qquad J_B - \frac{\omega^2}{2\pi\rho abc} = \frac{\theta}{b^2}, \qquad J_C = \frac{\theta}{c^2}, \tag{164}$$

with

$$\theta = 2\left(1 - \frac{\omega^2}{2\pi\rho}\right) \Big/ \left[abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\right].$$

For $e \neq 0$, we take

$$\int_0^\infty \frac{\phi_1}{\Delta} d\lambda = \theta P_0 \tag{165}$$

instead, where $\nabla^2 P_0 = 0$. In order to obtain a new bifurcation figure, we put

$$P = \xi(\alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2), \qquad P_0 = \frac{x}{a^2}\left(\alpha\frac{x^2}{a^4} + \beta\frac{y^2}{b^4} + \gamma\frac{z^2}{c^4} + \kappa\right).$$

Then we obtain the condition

$$\left.\begin{aligned}
\frac{\alpha}{2a^2}(c_2+c_3) - \frac{\beta}{2b^2}c_3 - \frac{\gamma}{2c^2}c_2 &= \theta\frac{\alpha}{a^6},\\
-\frac{3\alpha}{2a^2}c_3 + \frac{\beta}{2b^2}(3c_3+c_1) - \frac{\gamma}{2c^2}c_1 &= \theta\frac{\beta}{a^2b^4},\\
-\frac{3\alpha}{2a^2}c_2 - \frac{\beta}{2b^2}c_1 + \frac{\gamma}{2c^2}(3c_2+c_1) &= \theta\frac{\gamma}{a^2c^4},\\
\frac{3\alpha}{2a^2}\int_0^\infty \frac{\lambda d\lambda}{\Delta A^2} + \frac{\beta}{2b^2}\int_0^\infty \frac{\lambda d\lambda}{\Delta AB} + \frac{\gamma}{2c^2}\int_0^\infty \frac{\lambda d\lambda}{\Delta AC} + \kappa\int_0^\infty \frac{d\lambda}{\Delta A} &= \theta\frac{\kappa}{a^2},
\end{aligned}\right\} \tag{166}$$

where

$$c_1 = \int_0^\infty \frac{\lambda d\lambda}{\Delta ABC}, \qquad c_2 = \int_0^\infty \frac{\lambda d\lambda}{\Delta A^2 C}, \qquad c_3 = \int_0^\infty \frac{\lambda d\lambda}{\Delta A^2 B}.$$

From equation 166 we obtain, at first,

$$\frac{3\alpha}{a^4} + \frac{\beta}{b^4} + \frac{\gamma}{c^4} = 0,$$

which is the condition that P_0 should be harmonic. Next we obtain

$$a^2b^2c^2\left(\frac{3}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)(c_1c_2 + c_1c_3 + 3c_2c_3)$$
$$-\frac{2\theta}{a^2}[c_1(b^2+c^2) + c_2(3a^2+c^2) + c_3(3a^2+b^2)] + \left(\frac{2\theta}{a^2}\right)^2 = 0. \tag{167}$$

From equation 165 we obtain

$$\theta = \frac{\frac{1}{a^2} - \frac{1}{b^2} + \frac{3}{c^2}}{J_A - J_B + 3J_C}.$$

The ratio $a:b:c$ for the bifurcation figure is obtained from equation 167 with this value of θ.

Jeans then proceeded to prove the instability of a pear-shaped figure by computing the terms up to e^3, without heeding the convergence of the expansion. According to H. F. Baker (1920, 1926), the convergence of series expansions similar to Jeans' has been proved by Liapounov, and Jeans' series consists of terms in each of which he employed only a few first terms of the infinite series expansion in Lamé functions.

OTHER EQUILIBRIUM FIGURES

Jeans considered also tidally distorted masses and the problem of double stars after

Darwin 1910 (see also Glauert, 1915; Walton, 1914) and extended the study of the equilibrium figures of compressible fluid after Roche (Jeans, 1917, 1917a, 1919 (see also Lyttleton, 1953).

Problems of Saturn's rings as a liquid mass are treated by Kowalewski (1888); Poincaré (1885, 1885a); Levi-Civita (1908, 1912); Viterbi (1909, 1910); Tisserand (1880, 1889); and Klumpke (1895).

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CHAPTER VI

Theory of Liapounov

THE FUNCTIONAL EQUATION

The problem, how to determine equilibrium figures in the vicinity of the known ellipsoidal figures, is called *Tschebyscheff's problem.* Liapounov (1884, French translation 1904, and 1959) solved the problem and found the pear-shaped figure unstable, whereas Poincaré (1885) found it stable. So Liapounov further discussed the problem in great detail and published several voluminous papers entitled "Sur les figures d'équilibre peu différentes des ellipsoïdes d'une masse liquide homogène douée d'un mouvement de rotation," in the Mém. Acad. Sci. St. Pétersbourg, part I; 1906, pp. 1–225; part II, 1909, pp. 1–202; part III, 1912, pp. 1–229; part IV, 1914, pp. 1–112 (also 1905, 1908, 1909a, 1916, 1959).

He took as the three axes of the ellipsoid $\sqrt{\rho+1}$, $\sqrt{\rho+q}$, $\sqrt{\rho}$, $(q \leqslant 1)$, so that the surface is represented by

$$x=\sqrt{\rho+1}\sin\theta\cos\psi, \qquad y=\sqrt{\rho+q}\sin\theta\sin\psi, \qquad z=\sqrt{\rho}\cos\theta. \tag{168}$$

A distorted figure is represented by

$$x=\sqrt{\rho+\xi+1}\sin\theta\cos\psi, \qquad y=\sqrt{\rho+\xi+q}\sin\theta\sin\psi, \qquad z=\sqrt{\rho+\xi}\cos\theta. \tag{169}$$

The equation of the free surface is, with $\omega_0^2/2=\Omega_0$,

$$U+(\Omega_0+\eta)(\rho+\cos^2\psi+q\sin^2\psi+\xi)\sin^2\theta=\text{constant}, \tag{170}$$

where η is a function of a certain parameter α, and $\eta=0$ for $\alpha=0$, that is, for the original ellipsoid. The volume element $d\tau$ is expressed by

$$d\tau=\frac{H(\rho+\xi,\theta,\psi)}{2\Delta(\rho+\xi)}\,d\xi d\sigma,$$

where

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$$\left.\begin{aligned}\rho(\rho+q)\sin^2\theta\cos^2\psi+\rho(\rho+1)\sin^2\theta\sin^2\psi\\+(\rho+1)(\rho+q)\cos^2\theta=H(\rho,\theta,\psi),\\ \sqrt{\rho(\rho+1)(\rho+q)}=\Delta(\rho),\qquad \sin\theta d\theta d\psi=d\sigma.\end{aligned}\right\}\tag{171}$$

With $r=D(\rho+\xi,\ \rho+\xi')$, the distance between two points $(\rho+\xi,\ \theta,\ \psi)$ and $(\rho+\xi',\ \theta',\ \psi')$, we have

$$\begin{aligned}U&=\frac{1}{2\pi}\int d\sigma'\int_{-\rho}^{\xi'}\frac{H(\rho+\xi',\ \theta',\ \psi')}{r\Delta(\rho+\xi')}d\xi'\\&=\Phi(\rho+\xi)+S,\end{aligned}$$

where

$$\begin{aligned}\Phi(\rho)&=\frac{1}{2\pi}\int d\sigma'\int_0^{\rho}\frac{H(\rho',\ \theta',\ \psi')}{D(\rho,\ \rho')\Delta(\rho')}d\rho',\\ S&=\frac{1}{2\pi}\int d\sigma'\int_{\xi}^{\xi'}\frac{H(\rho+\xi',\ \theta',\ \psi')}{r\Delta(\rho+\xi')}d\xi';\end{aligned}$$

$\pi\Phi(\rho)$ is the potential of an ellipsoid with three axes $\sqrt{\rho+1}$, $\sqrt{\rho+q}$, $\sqrt{\rho}$ and it is known that

$$\Phi(\rho)=\Delta(\rho)\int_{\rho}^{\infty}\left(1-\frac{\rho+1}{t+1}\sin^2\theta\cos^2\psi-\frac{\rho+q}{t+q}\sin^2\theta\sin^2\psi-\frac{\rho}{t}\cos^2\theta\right)\frac{dt}{\Delta(t)}.$$

So far as $|\zeta|<\rho$, we can develop $\Phi(\rho+\zeta)$ in powers of ζ:

$$\Phi(\rho+\zeta)=\Phi(\rho)+\Phi'(\rho)\zeta+\frac{1}{1\cdot 2}\Phi''(\rho)\zeta^2+\ldots.$$

Denote the angle between $(\theta,\ \psi)$ and $(\theta',\ \psi')$ by φ; then, so far as

$$\ell+g<1,\qquad \left|\frac{\zeta}{\rho}\right|<\ell,\qquad \frac{|\zeta'-\zeta|}{2\rho\sqrt{2(1-\cos\varphi)}}<g,$$

we can develop $S(\epsilon)$ uniformly for all values of ξ, θ', ψ' in the form

$$S(\epsilon)=S_1\epsilon+S_2\epsilon^2+\ldots,$$

or, putting $\epsilon=1$,

$$S=S_1+S_2+\ldots,$$

where

$$\frac{\epsilon}{2\pi}\int d\sigma'\int_{\zeta}^{\zeta'}\frac{H(\rho+\epsilon\xi,\ \theta',\ \psi')}{D(u+\epsilon\zeta,\ \rho+\epsilon\xi)\Delta(\rho+\epsilon\xi)}d\xi=S(u,\ \epsilon),$$

$$S(\epsilon)=\lim_{u=\rho}S(u,\ \epsilon).$$

Hence

$$U = U_0 + U_1 + U_2 + \ldots,$$

$$U_0 = \Phi(\rho),$$

$$U_n = \frac{1}{n!}\Phi^{(n)}(\rho)\zeta^n + S_n \qquad (n > 0),$$

where

$$S_n = \frac{1}{2\pi}\lim_{u=\rho}\sum_{i+j=n-1}\frac{\zeta^i}{i!(j+1)!}\left(\frac{\partial^{n-j}}{\partial u^i \partial v^j}\int\frac{H(v,\theta',\psi')\zeta^{j+1}}{\Delta(v)D(u,v)}d\sigma'\right)_{v=p} - \frac{1}{n!}\Phi^{(n)}(\rho)\zeta^n.$$

For $n=1$, our expression $[\partial\Phi(u,\rho)/\partial u]\zeta$ should be added on the right-hand side. Let

$$\Delta(\rho)\int_\rho^\infty \frac{dt}{(t+1)\Delta(t)} = L,$$

$$\Delta(\rho)\int_\rho^\infty \frac{dt}{(t+q)\Delta(t)} = M,$$

$$\Delta(\rho)\int_\rho^\infty \frac{dt}{t\Delta(t)} = N;$$

then,

$$\frac{\partial\Phi(u,\rho)}{\partial u} = -L\sin^2\theta\cos^2\psi - M\sin^2\theta\sin^2\psi - N\cos^2\theta$$

for $u < \rho$. The condition necessary for the original ellipsoid to be an equilibrium figure is

$$(\rho+1)(L-\Omega_0) = (\rho+q)(M-\Omega_0) = \rho N. \tag{172}$$

Let

$$\Delta(\rho) = \Delta, \qquad H(\rho,\theta,\psi) = H, \qquad H(\rho,\theta',\psi') = H', \qquad \frac{1}{2}\rho\int_\rho^\infty \frac{dt}{t\Delta(t)} = R, \tag{173}$$

and $D(\rho,\rho) = D$, the distance between two points (ρ,ψ,θ) and (ρ,ψ',θ'); then,

$$R = \frac{\rho}{2\Delta}N, \qquad \frac{\partial\Phi(u,\rho)}{\partial u} = -\frac{2}{\Delta}RH - \Omega_0\sin^2\theta,$$

$$U_1 = \frac{1}{2\pi\Delta}\int\frac{H'\zeta' d\sigma'}{D} - \left(\frac{2}{\Delta}RH + \Omega_0\sin^2\theta\right).$$

Substitute these in equation 170 and note that, the ellipsoid being an equilibrium figure, we have

$$U_0 + \Omega_0(\rho + \cos^2\psi + q\sin^2\psi)\sin^2\theta = \text{constant}.$$

Then we obtain the fundamental functional equation:

$$RH\zeta-\frac{1}{4\pi}\int\frac{H'\zeta'd\sigma'}{D}=\frac{\Delta}{2}W+\text{constant},$$

$$W=\eta(\rho+\cos^2\psi+q\sin^2\psi+\zeta)\sin^2\theta+U_2+U_3+\ldots \tag{174}$$

The volume should be constant, and the center of mass of the new figure should be at the origin; the principal axes of inertia should be on the x- and y-axes. These conditions are expressed by

$$\int d\sigma\int_0^\zeta\frac{H(\rho+\xi,\theta,\psi)}{\Delta(\rho+\xi)}d\xi=0,\qquad\int\cos\theta\, d\sigma\int_0^\zeta\frac{H(\rho+\xi,\theta,\psi)}{\sqrt{(\rho+\xi+1)(\rho+\xi+q)}}d\xi=0,$$

$$\int\sin^2\theta\sin 2\psi d\sigma\int_0^\zeta\frac{H(\rho+\xi,\theta,\psi)}{\sqrt{\rho+\xi}}d\xi=0. \tag{175}$$

The latter two equations can be written

$$\int H\zeta\cos\theta d\sigma=0,\qquad\int H\zeta\sin^2\theta\cos 2\psi d\sigma=0.$$

LAMÉ FUNCTIONS

Put

$$\sqrt{1-\mu^2}\sqrt{1-\nu^2}=\sqrt{1-q}\sin\theta\cos\psi,$$
$$\sqrt{1-\mu^2}\sqrt{\nu^2-q}=\sqrt{q(1-q)}\sin\theta\sin\psi,$$
$$\mu\nu=\sqrt{q}\cos\theta.$$

Lamé functions are the solutions in the form of integral polynomials in x, $\sqrt{x^2-1}$, $\sqrt{x^2-q}$ of

$$\sqrt{(x^2-1)(x^2-q)}\frac{d}{dx}\left[\sqrt{(x^2-1)(x^2-q)}\frac{dy}{dx}\right]+[\beta-n(n+1)x^2]y=0.$$

For each value of β such as $\beta_{n,0}>\beta_{n,1}>\ldots>\beta_{n,2n}$, there corresponds a solution $E_{n,0}(x)$, $\ldots$, $E_{n,2n}(x)$

$$s\equiv 0\ (\text{mod. }4): E_{ns}(x)=P,$$
$$s\equiv 1\ (\text{mod. }4): E_{ns}(x)=P\sqrt{x^2-q},$$
$$s\equiv 2\ (\text{mod. }4): E_{ns}(x)=P\sqrt{x^2-1},$$
$$s\equiv 3\ (\text{mod. }4): E_{ns}(x)=P\sqrt{x^2-1}\sqrt{x^2-q},$$

where

$$P=c_0x^m-c_1x^{m-2}+\ldots\qquad(m=n,\ n-1,\ n-2).$$

For example:

$$n=0 : E_{00}(x)=c_0;$$
$$n=1 : E_{10}(x)=c_0 x, \quad E_{11}(x)=c_0\sqrt{x^2-q}, \quad E_{12}(x)=c_0\sqrt{x^2-1};$$
$$n=2 : E_{20}(x)=c_0(x^2-k'), \quad E_{21}(x)=c_0 x\sqrt{x^2-q}, \quad E_{22}(x)=c_0 x\sqrt{x^2-1},$$
$$E_{23}(x)=c_0\sqrt{x^2-1}\,\sqrt{x^2-q}, \quad E_{24}(x)=c_0(x^2-k''),$$

where $k''>k'$ are the roots of $3k^2-2(1+q)k+q=0$.

Lamé functions of the second kind are

$$F_{n,s}(x)=(2n+1)E_{ns}(x)\int_x^\infty \frac{dx}{[E_{ns}(x)]^2\sqrt{(x^2-1)(x^2-q)}}.$$

Write

$$E_{ns}(\sqrt{-u})=\mathsf{E}_{ns}(u),\ \mathsf{F}_{ns}(u)=\frac{2n+1}{2}\,\mathsf{E}_{ns}(u)\int_u^\infty \frac{du}{[\mathsf{E}_{ns}(u)]^2\sqrt{u(u+1)(u+q)}}; \tag{176}$$

then equation 173 is written

$$R=\frac{1}{3}\,\mathsf{E}_{10}\mathsf{F}_{10}. \tag{177}$$

Eliminating Ω_0 from equation 172, we obtain the relation between ρ and q for a Jacobi ellipsoid

$$(\rho+1)(\rho+q)(M-L)=(1-q)\rho N,$$

or

$$\rho\int_\rho^\infty \frac{dt}{t\Delta(t)}-(\rho+1)(\rho+q)\int_\rho^\infty \frac{dt}{(t+1)(t+q)\Delta(t)}=0,$$

or

$$\frac{1}{3}\,\mathsf{E}_{10}\mathsf{F}_{10}-\frac{1}{5}\,\mathsf{E}_{23}\mathsf{F}_{23}=0. \tag{178}$$

We have relations of the form

$$\left.\begin{aligned}\int\frac{[E_{ns}(\mu)E_{ns}(\nu)]^2}{(\rho+\mu^2)(\rho+\nu^2)}\,d\sigma&=\gamma_{ns}\frac{\mathsf{E}_{ns}\mathsf{F}_{ns}}{\Delta},\\ \Delta&=\sqrt{\rho(\rho+1)(\rho+q)},\\ \gamma_{ns}&=\int[E_{ns}(\mu)E_{ns}(\nu)]^2d\sigma;\end{aligned}\right\} \tag{179}$$

$$\begin{aligned}\int\frac{E_{ns}(\mu')E_{ns}(\nu')}{D(u,v)}\,d\sigma'&=\frac{4\pi}{2n+1}\,\mathsf{E}_{ns}(u)\mathsf{F}_{ns}(v)E_{ns}(\mu)E_{ns}(\nu), \qquad (u\leqslant v)\\ &=\frac{4\pi}{2n+1}\,\mathsf{E}_{ns}(v)\mathsf{F}_{ns}(u)E_{ns}(\mu)E_{ns}(\nu), \qquad (u\geqslant v).\end{aligned} \tag{179a}$$

For $q \to 1$, $\nu^2 > \mu^2$, we have, according as $s=2k$ or $s=2k-1$: $\lim E_{ns}(\mu) = P_{nk}(\cos\theta)$, $\lim E_{n,2k}(\nu) = \cos k\psi$, or, $\lim E_{n,2k-1}(\nu) = \sin k\psi$, $\lim \mathsf{E}_{ns}(u) = \mathsf{P}_{ns}(u)$, $\lim \mathsf{F}_{ns}(u) = \mathsf{Q}_{nk}(u)$, where

$$\mathsf{P}_{n,k}(u) = \frac{(\sqrt{u+1})^k}{2\cdot 4\cdot \ldots 2n}\left[\frac{d^{n+k}(x^2+1)^n}{dx^{n+k}}\right]_{x=\sqrt{u}}$$

$$\mathsf{Q}_{nk}(u) = \frac{2n+1}{2}\mathsf{P}_{nk}(u)\int_u^\infty \frac{du}{[\mathsf{P}_{nk}(u)]^2(u+1)\sqrt{u}}.$$

Note that

$$P_n(x) = \frac{1}{2\cdot 4\cdot \ldots 2n}\,\frac{d^n(x^2-1)^n}{dx^n},$$

$$P_{n,k}(x) = (\sqrt{1-x^2})^k\frac{d^kP_n(x)}{dx^k}.$$

BIFURCATION FIGURES

From equation 174 we proceed by successive approximations for calculating the unknown quantity ζ. In our first approximation, we consider

$$RHz - \frac{1}{4\pi}\int\frac{H'z'd\sigma'}{D} = Z, \tag{180}$$

where

$$R = (\rho/2)\int_\rho^\infty \frac{dt}{t\Delta(t)} = \frac{1}{3}\,\mathsf{E}_{10}\mathsf{F}_{10},$$

and Z is known on the surface. Multiply by $E_{ns}(\mu)\,E_{ns}(\nu)\,d\sigma$ and integrate over the surface. From equation 179a, we obtain

$$T_{ns}\int HzE_{ns}(\mu)E_{ns}(\nu)\,d\sigma = \int ZE_{ns}(\mu)E_{ns}(\nu)\,d\sigma, \tag{181}$$

$$T_{ns} = (1/3)\,\mathsf{E}_{10}\mathsf{F}_{10} - \frac{1}{2n+1}\mathsf{E}_{ns}\mathsf{F}_{ns}. \tag{182}$$

For the values of n, s satisfying $T_{ns}=0$, we have $\int ZE_{ns}(\mu)E_{ns}(\nu)d\sigma = 0$; $n=1$, $s=0$ is one of such cases. For $n=2$, $s=3$, we have the Jacobi ellipsoid, and $T_{2,3}=0$. For such special values of m, r which make $T_{mr}=0$, we have $\int ZE_{mr}(\mu)E_{mr}(\nu)d\sigma = 0$, and

$$\int HzE_{mr}(\mu)E_{mr}(\nu)d\sigma \tag{183}$$

can take an arbitrary value, while for any other pair of values n, s,

$$\int HzE_{ns}(\mu)E_{ns}(\nu)d\sigma \tag{184}$$

takes a definite value; thus z is determined.

Take a very small parameter α; for $\alpha = 0$ the figure is supposed to reduce to the ellipsoid from which we start. Assume that

$$\left|\frac{\zeta}{\rho}\right| < l, \qquad \frac{|\zeta' - \zeta|}{2\rho\sqrt{2(1-\cos\varphi)}} < g, \tag{185}$$

where l, g are constants, which can be taken as small as we please as long as α is sufficiently small (the left-hand sides vanish for $\alpha=1$). It is proved after a long, laborious computation that g/l is a fixed number.

Now our problem is to see whether there exists a new equilibrium figure slightly different from the Maclaurin or the Jacobi ellipsoids under the conditions of equation 185. Denote the ellipsoid from which we start by E_0. Let the new figure be of the same volume, its center of mass be at the origin, and its principal axes of inertia coincide with E_0; that is, equations 175 are satisfied. If the center of mass is at the origin, we have

$$\int \cos\theta d\sigma \int_0^\zeta \frac{H(\rho+\xi,\theta,\psi)}{\sqrt{(\rho+\xi+1)(\rho+\xi+q)}} d\xi = 0. \tag{186}$$

If the principal axes of inertia coincide with the coordinate axes, we have

$$\int \sin^2\theta \sin 2\psi d\sigma \int_0^\zeta \frac{H(\rho+\xi,\theta,\psi)}{\sqrt{\rho+\xi}} d\xi = 0 \tag{187}$$

for the Jacobi ellipsoid, and

$$\left.\begin{aligned} &\int H\zeta P_{m,k}(\cos\theta)\sin k\psi d\sigma = 0, \\ &\text{or} \\ &\int H\zeta E_{m,2k-1}(\mu)E_{m,2k-1}(\nu)d\sigma = 0 \qquad (k>0) \end{aligned}\right\} \tag{188}$$

for the Maclaurin spheroid. Expanding the integrals in equation 175 in powers of ζ, we obtain

$$\int H\zeta d\sigma = I_0, \qquad \int H\zeta\cos\theta d\sigma = I_1, \qquad \int H\zeta\sin^2\theta\sin 2\psi d\sigma = I_2, \tag{189}$$

or

$$\int H\zeta E_{10}(\mu)E_{10}(\nu)d\sigma = I_1, \qquad \int H\zeta E_{23}(\mu)E_{23}(\nu)d\sigma = I_2, \tag{190}$$

where I_0, I_1, I_2 contain terms of degree higher than 2 with regard to ζ.

Suppose that the solution of the fundamental equation 174 is

$$\zeta = z + \frac{1}{H}\sum a_{mr}E_{mr}(\mu)E_{mr}(\nu), \tag{191}$$

where a_{mr} are suitably chosen constants, and z satisfies

$$\int HzE_{mr}(\mu)E_{mr}(\nu)d\sigma = 0.$$

Suppose further that $T_{mr}=0$. For such pairs of m, r, we see that the solution z of equation 174 satisfies

$$|z| < \frac{L_0+ML}{\rho(\rho+q)},$$

where L_0, M, L are sufficiently small constants, tending to 0 with l^2, or with α. We seek for the values of a_{mr}/l that do not tend to zero with α, except a_{10}, a_{23}; note that T_{00}, T_{11}, T_{12} cannot become zero.

Liapounov at first proved the theorem that an equilibrium figure which can be derived from an ellipsoid has at least two symmetry planes.

SUCCESSIVE APPROXIMATIONS

We start from E_0 with Ω_0 and consider an ellipsoid E with $\Omega_0+\eta$. We seek equilibrium figures with the same volume as E, the same center of mass, and the same principal axes of inertia. The conditions are given in equations 186, 187, and 188. But under this condition we cannot have

$$\int H\zeta E_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma=0.$$

Write

$$\left.\begin{aligned} &\int H\zeta E_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma=\gamma\alpha,\\ &\gamma=\int [E_{m,2k}(\mu)E_{m,2k}(\nu)]^2 d\sigma. \end{aligned}\right\} \tag{192}$$

This gives our parameter α.

Then,

$$H\zeta=\alpha E_{m,2k}(\mu)E_{m,2k}(\nu)+Hz, \tag{193}$$

$$\int HzE_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma=0. \tag{194}$$

The fundamental equation 174 can be written

$$\left.\begin{aligned} &RHz-\frac{1}{4\pi}\int\frac{H'z'd\sigma'}{D}=Z+\text{constant},\\ &Z=\frac{\Delta}{2}W-\alpha T_{m,2k}E_{m,2k}(\mu)E_{m,2k}(\nu),\\ &W=U_2+U_3+\ .\ .\ .\ . \end{aligned}\right\} \tag{195}$$

It can be proved that

$$\frac{\alpha E_{m,2k}(\mu)E_{m,2k}(\nu)}{H}$$

can be taken as a first approximation to ζ.

If we stop at the first-order approximation with regard to ζ, the equilibrium figure we seek is

$$\frac{x^2}{\rho+1}+\frac{y^2}{\rho+q}+\frac{z^2}{\rho}=1+\frac{\alpha}{\Delta^2}E_{m,2k}(\mu)E_{m,2k}(\nu); \tag{196}$$

$E_{m,2k}(\mu)E_{m,2k}(\nu)$ is of degree m with regard to $\sin\theta\cos\psi$, $\sin\theta\sin\psi$, $\cos\theta$. It is recalled that the expansions employed by Jeans are of such nature as Baker remarked.

The equilibrium figures are referred to the ellipsoid E in the above. Now we must refer them to the original ellipsoid E_0. At first the volumes of E and E_0 should be equal; that is,

$$(1+\epsilon)^3=\frac{\rho_0(\rho_0+1)(\rho_0+q_0)}{\rho(\rho+1)(\rho+q)}. \tag{197}$$

We have further

$$\left.\begin{aligned}\sqrt{\rho_0+1+\zeta_0}\,\sin\theta_0\cos\psi_0&=\sqrt{1+\epsilon}\,\sqrt{\rho+1+\zeta}\,\sin\theta\cos\psi,\\ \sqrt{\rho_0+q+\zeta_0}\,\sin\theta_0\sin\psi_0&=\sqrt{1+\epsilon}\,\sqrt{\rho+q+\zeta}\,\sin\theta\sin\psi,\\ \sqrt{\rho_0+\zeta_0}\,\cos\theta_0&=\sqrt{1+\epsilon}\,\sqrt{\rho+\zeta}\,\cos\theta.\end{aligned}\right\} \tag{198}$$

We have supposed that, as α and η tend to zero, the equilibrium figure tends to E_0; also ϵ tends to zero with α and η.

The important part of Liapounov's work is the proof of the convergence of the process of successive approximations.

Put

$$\zeta=\sum\zeta_{rs}\alpha^r\eta^s=\zeta_{10}\alpha+\zeta_{01}\eta+\ldots; \tag{199}$$

where ζ is a function of θ and ψ. Put $|\zeta_{rs}|<\rho l_{rs}$. Liapounov proved the convergence of $\Sigma l_{rs}\alpha^r\eta^s$ for sufficiently small $|\alpha|$ and $|\eta|$. Put $|\zeta_{rs}'-\zeta_{rs}|<2\rho g_{rs}\sqrt{2(1-\cos\varphi)}$. He also proved the convergence of $\Sigma g_{rs}\alpha^r\eta^s$. Then he constructed a majorant series for $|U-U_0|$ in the form of the expansion in powers of l and g of the function

$$2(\rho+1)\left[\frac{(1+l)^3}{(1-l)^2}-1+\frac{2(1+l)^2}{\sqrt{(1-l)^2+g^2}}\frac{l}{1-l}\right],$$

where

$$l=\sum l_{rs}|\alpha|^r|\eta|^s,\qquad g=\sum g_{rs}|\alpha|^r|\eta|^s,\qquad l+g<1.$$

He also constructed majorant series for W and

$$\frac{W'-W}{2\rho\sqrt{2(1-\cos\varphi)}},$$

and proved the convergence of the series he employed.

Now we compare the parameters ρ, q of an ellipsoid satisfying $T_{m,2k}=0\,(m-k=\text{even})$ to those parameters for E_o, by taking ζ as the solution of equation 174 with the conditions of equations 186, 187, and 188. From the fundamental equation, together with $T_{m,2k}=0$, we obtain

$$\int WE_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma=0. \tag{200}$$

For the moment, let

$$\int WE_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma=\gamma\mathsf{A};$$

then we have

$$RH\zeta-\frac{1}{4\pi}\int\frac{H'\zeta'd\sigma'}{D}=\frac{\Delta}{2}\left[W-\mathsf{A}E_{m,2k}(\mu)E_{m,2k}(\nu)\right]+\text{constant}. \tag{201}$$

Write equation 199 in the form

$$\zeta=\sum_{n=1}^{\infty}\zeta_n \quad\text{and}\quad W=\sum_{n=1}^{\infty}W_n;$$

then equation 201 takes the form

$$RH\zeta_n-\frac{1}{4\pi}\int\frac{H'\zeta_n'd\sigma'}{D}=\frac{\Delta}{2}\left(W_n-\mathsf{A}_nE_{m,2k}(\mu)E_{m,2k}(\nu)\right)+\text{constant},$$
$$\mathsf{A}_n=\frac{1}{\gamma}\int W_nE_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma, \tag{202}$$

and the conditions are

$$\left.\begin{aligned}\int H\zeta_nE_{10}(\mu)E_{10}(\nu)\,d\sigma&=0,\\ \int H\zeta_nE_{23}(\mu)E_{23}(\nu)\,d\sigma&=0,\\ \int H\zeta_nE_{m,\,2k-1}(\mu)E_{m,\,2k-1}(\nu)\,d\sigma&=0,\\ \int H\zeta_n\,d\sigma&=I_n.\end{aligned}\right\} \tag{203}$$

From $U_n=U_{n,\,n}+U_{n,\,n+1}+\ldots$, we obtain the expansion of $W=\Sigma W_n$, where

$$\left.\begin{aligned}W_1&=\eta(\rho+\cos^2\psi+q\sin^2\psi)\sin^2\theta,\\ W_2&=\eta\zeta_1\sin^2\theta+U_{2,\,2},\\ &\ldots,\\ W_n&=\eta\zeta_{n-1}\sin^2\theta+U_{2,\,n}+U_{3,\,n}+\ldots+U_{n,\,n}.\end{aligned}\right\} \tag{204}$$

W_1 is a function of θ, ψ; W_i contains ζ_s, $s<i$. If we compute ζ_s, $s<i$, successively, we obtain W_i as a function of θ, ψ. Similarly, I_i contains ζ_s, $s<i$, $I_1=0$. Hence we can compute ζ_1, ζ_2, . . . successively from

$$\left.\begin{aligned}RH\zeta_n-\frac{1}{4\pi}\int\frac{H'\zeta_n'd\sigma'}{D}&=Z+\text{constant},\\ Z&=\frac{\Delta}{2}\left[W_n-\mathsf{A}_nE_{m,2k}(\mu)E_{m,2k}(\nu)\right].\end{aligned}\right\} \tag{205}$$

This equation has the form of equation 180.

The next step is to make it satisfy

$$\int H\zeta_n E_{m,2k}(\mu)E_{m,2k}(\nu)\,d\sigma=0, \qquad (n>1).$$

From this equation, the parameter α is determined. In order to satisfy this equation, the equation

$$\int ZE_{rs}(\mu)E_{rs}(\nu)\,d\sigma=0 \tag{206}$$

should be satisfied by the pairs $(1,0)$, $(2,3)$, $(m, 2k-1)$, $(m, 2k)$ for (r, s), owing to the condition $T_{rs}=0$ for such pairs, which are satisfied actually. Hence ζ_n is determined. For the fourth pair, we know that it is satisfied by equation 202.

After a long discussion, Liapounov proved the convergence of the series $\zeta=\Sigma\zeta_n$. From the second step, we see $\mathsf{A}=0$ from equation 200 or

$$\sum \mathsf{A}_{rs}\alpha^r\eta^s=0.$$

By the definition of W_n in equation 204, we see that there is no nonellipsoidal equilibrium figure starting from E_0 either for $m=2$, $k=0$, or for $m=k=2$. Thus Liapounov reached the conclusion that, in order that there may exist a nonellipsoidal equilibrium figure differing from E_0 as much as we please (where E_0 may be a Maclaurin spheroid or a Jacobi ellipsoid), it is necessary and sufficient to have

$$\frac{1}{3}\mathsf{E}_{10}\mathsf{F}_{10}-\frac{1}{2m+1}\mathsf{E}_{m,2k}\mathsf{F}_{m,2k}=0. \tag{207}$$

Here $m>2$, and $m-k$ is even; $m-k$ cannot be zero for a Jacobi ellipsoid. Equation 207 determines uniquely the starting ellipsoid E_0.

Such an equilibrium figure has a symmetry plane perpendicular to the rotation axis and at least one more symmetry plane through the rotation axis. If E_0 is Maclaurin's, the equilibrium figure is one of revolution for $k=0$ and has k symmetry planes through the rotation axis for $k\neq 0$. If E_0 is Jacobi's, it has two symmetry planes through the rotation axis for m even but only one for m odd.

Let z be the rotation axis and xz be the symmetry plane. If E_0 is Jacobi's, the surface of the equilibrium figure is represented by

$$x=\sqrt{\rho+1+\zeta}\,\sin\theta\cos\psi=\sqrt{\rho+1+\zeta}\,\frac{\sqrt{1-\mu^2}\,\sqrt{1-\nu^2}}{\sqrt{1-q}},$$

$$y=\sqrt{\rho+q+\zeta}\,\sin\theta\sin\psi=\sqrt{\rho+q+\zeta}\,\frac{\sqrt{q-\mu^2}\,\sqrt{\nu^2-q}}{\sqrt{q(1-q)}},$$

$$z=\sqrt{\rho+\zeta}\,\cos\theta \qquad =\sqrt{\rho+\zeta}\,\frac{\mu\nu}{\sqrt{q}},$$

where $\sqrt{\rho+1}$, $\sqrt{\rho+q}$, $\sqrt{\rho}$ are the axes of E_0, and ρ, q are determined by equation 207. Put

$$(\rho+\mu^2)(\rho+\nu^2)\zeta=\alpha E_{m,2k}(\mu)E_{m,2k}(\nu)+\xi;$$

then ξ should satisfy

$$\int_0^{2\pi} d\psi \int_0^{\pi} \xi E_{m,\,2k}(\mu)E_{m,\,2k}(\nu)\sin\theta d\theta=0,$$

with $\alpha=$ constant for any value of α. Then there exists one and only one equilibrium figure with the same volume as E_0 such that the greater of $|\zeta|$ and

$$\left|\frac{\zeta'-\zeta}{\sqrt{1-\cos\varphi}}\right|$$

is sufficiently small; ξ is expressed by a positive integral power series of α, and the series is absolutely and uniformly convergent for all values of θ, ψ as long as $|\alpha|$ is sufficiently small.

DERIVED FIGURES

Part II of the work by Liapounov contains the discussion in particular of the figures derivable from Maclaurin spheroids.

$$\mathsf{A}=\sum \mathsf{A}_{rs}\alpha^r\eta^s.$$

In the equations he wrote $\mathsf{A}_{i0}=A_i$, $\mathsf{A}_{11}=B$, $\mathsf{A}_{0i}=C_i$, $\mathsf{A}_{21}\alpha+\mathsf{A}_{12}\eta+\mathsf{A}_{31}\alpha^2+\mathsf{A}_{22}\alpha\eta+\ldots=S$; $S=0$ for $\alpha=\eta=0$, so that $A_2\alpha^2+A_3\alpha^3+\ldots+(B+S)\alpha\eta+C_3\eta^3+\ldots=0$. At first he proved that $B\neq 0$. If all $A_i=0$, then there would be a solution such that $\eta=0$; that is, the new figure would rotate with the same angular velocity, but it was proved that there are some A_i which are not zero. Suppose that $A=A_{\lambda+1}\neq 0$ for such a figure; then $\eta=-(A/B)\alpha^\lambda+\ldots$. If λ is even, then A/B can have two signs, but if λ is odd, A/B has only one sign.

If the figure is not one of revolution, then $A_2=0$. For $m=k=2$, that is, for the junction of Jacobi series and Maclaurin series, we have $A_3\neq 0$. A similar result also holds for $m=k=3$. It is proved after a long series of computation that there is no pair (m, k) for which $A_3=0$. The result is: for $k=0$, the angular velocity is expressed in the transition from an ellipsoid of revolution to an equilibrium figure of revolution in the form

$$\eta=\eta_1\alpha+\eta_2\alpha^2+\ldots\quad(\eta_1\neq 0),$$

and in the transition to an equilibrium figure not of revolution in the form

$$\eta=\eta_2\alpha^2+\eta_4\alpha^4+\ldots\quad(\eta_2\neq 0).$$

This ζ is developed in a positive integral power of η in the first case and in a positive integral power of $\sqrt{\eta}$ in the second case. There is only one ζ in the first case; there are two values of ζ, but with the same figure, in the second case. Thus there exists one and only one distinct equilibrium figure for a given value of η with a fixed sign of η for $k\neq 0$; that is, $\eta<0$ for $m=k=2$. The angular velocity decreases in passing from a Maclaurin spheroid to a Jacobi ellipsoid. A similar situation holds also for

$$m=k=3, \qquad \eta_2=-\frac{A_3}{B}, \qquad A_3>0,\ B>0.$$

In the other cases, $B<0$; that is, η has the same sign as A_3.

SINGULAR ELLIPSOID

Part III is devoted to the discussion of the figures derivable from a Jacobi ellipsoid. The solution for ρ and q of the equations $T_{23}=0$, $T_m \equiv T_{m,\,2m}=0$ $(m=3, 4, \ldots)$ is uniquely determined for each value of m; $m=3$ corresponds to a pear-shaped figure. It is proved that $T_{3,\,6}<T_{4,\,8}<T_{5,\,10}< \ldots .$ Liapounov then computed exactly to four decimal places.

Let

$$R=\frac{1}{3}\,\mathsf{E}_{10}\mathsf{F}_{10}=\frac{1}{2}\,\rho\int_\rho^\infty \frac{dt}{t\Delta(t)}, \qquad Q=\frac{1}{5}\,\mathsf{E}_{23}\mathsf{F}_{23}=\frac{1}{2}\,(\rho+1)(\rho+q)\int_\rho^\infty \frac{dt}{(t+1)(t+q)\Delta(t)};$$

then $T_{23}=0$ is written $R-Q=0$, where

$$\sqrt{\kappa+1}\,R=\int_0^1 \frac{z^2dz}{\sqrt{\rho+z^2}\,\sqrt{1-\lambda(1-z^2)}}, \qquad \sqrt{\kappa+1}\,Q=(\rho+1)\int_0^1 \frac{z^4dz}{(\rho+z^2)^{3/2}[1-\lambda(1-z^2)]^{3/2}},$$

with

$$t=\frac{\rho}{z^2}, \qquad \frac{q}{\rho}=\kappa, \qquad \frac{\kappa}{\kappa+1}=\lambda.$$

Furthermore,

$$\frac{\Delta}{\rho}R=a_0+\frac{1}{2}\,a_1\lambda+\frac{1.3}{2.4}\,a_2\lambda^2+\ldots, \qquad \frac{\Delta}{\rho}Q=b_0+\frac{3}{2}\,b_1\lambda+\frac{3.5}{2.4}\,b_2\lambda^2+\ldots, \qquad \Delta=\Delta(\rho),$$

with

$$\sqrt{\rho+1}\int_0^1 \frac{(1-z^2)^nz^2dz}{\sqrt{\rho+z^2}}=a_n, \qquad (\rho+1)^{3/2}\int_0^1 \frac{(1-z^2)^nz^4dz}{(\rho+z^2)^{3/2}}=b_n;$$

$R-Q=0$ is written $\lambda f(\lambda)=c$, with $a_0-b_0=c$, $a_n-b_n=c_n$,

$$\frac{2n+1}{2n}\,b_n-\frac{1}{2n}\,a_n=l_{n-1} \qquad (n=1, 2, \ldots), \qquad l_0+\frac{3}{2}\,l_1\lambda+\frac{3.5}{2.4}\,l_2\lambda^2+\ldots=f(\lambda).$$

Liapounov computed a_n, b_n as far as $n=24$, by estimating the errors of computation at each step. Thus he computed for $m=3$ the values of

$$0.362648151\left\{\begin{matrix}2\\1\end{matrix}\right. <\lambda< 0.362655458\left\{\begin{matrix}1\\0\end{matrix}\right.$$

and the values of the three axes of the ellipsoid with the same volume:

	Darwin's value
$1.8856\left\{\begin{matrix}27\\52\end{matrix}\right. < \alpha < 1.8856\left\{\begin{matrix}28\\27\end{matrix}\right.$	1.885827
$0.8150\left\{\begin{matrix}40\\31\end{matrix}\right. < \beta < 0.8150\left\{\begin{matrix}40\\39\end{matrix}\right.$	0.814975
$0.6506\left\{\begin{matrix}78\\74\end{matrix}\right. < \gamma < 0.6506\left\{\begin{matrix}78\\77\end{matrix}\right.$	0.650659

Similarly he computed these values for $m=4$.

The computation of $T_m=0$ is very complicated:

$$T_m=R-\frac{1}{2m+1}\mathsf{E}_m\mathsf{F}_m, \qquad \frac{\mathsf{EF}}{\Delta}=\frac{1}{8}\int_\rho^\infty \frac{[E(\mu)E(\nu)]^2}{(\rho+\mu^2)(\rho+\nu^2)}\,d\sigma.$$

Put

$$\mathsf{M}=\sqrt{(1-\mu^2)(q-\mu^2)}, \qquad \mathsf{N}=\sqrt{(1-\nu^2)(\nu^2-q)},$$

$$R=\frac{1}{3}\mathsf{E}_{10}\mathsf{F}_{10}=\frac{\Delta}{4q\pi}\int_0^1 \frac{\mu^2\nu^2 d\sigma}{(\rho+\mu^2)(\rho+\nu^2)},$$

$$Q=\frac{3\Delta}{4\pi q(1-q)^2}\int_0^1 \frac{\mathsf{M}^2\mathsf{N}^2 d\sigma}{(\rho+\mu^2)(\rho+\nu^2)}, \qquad d\sigma=\frac{(\nu^2-\mu^2)d\mu d\nu}{\mathsf{MN}};$$

and

$$\mathsf{E}=(\rho+h_1)(\rho+h_2)\ \ldots\ (\rho+h_n),$$

$$P=\sum_{i=1}^n \frac{1}{2h_i(1-h_i)(h_i-q)[\mathsf{E}'(-h_i)]^2}\frac{\mathsf{E}}{\rho+h_i},$$

$$g=\frac{3}{4q}\sum_{i=1}^n \frac{1}{h_i[\mathsf{E}'(-h_i)]^2}, \qquad \mathscr{g}=\frac{1}{4q}\sum_{i=1}^n \frac{1}{(1-h_i)(h_i-q)[\mathsf{E}'(-h_i)]^2}, \qquad \text{for } m=2n;$$

$$\mathsf{E}=\sqrt{\rho+1}(\rho+h_1)\ \ldots\ (\rho+h_n),$$

$$\Phi(\rho)=(\rho+1)(\rho+h_1)\ \ldots\ (\rho+h_n)=\sqrt{\rho+1}\,\mathsf{E},$$

$$P=\sum_{i=1}^n \frac{1}{2h_i(h_i-q)[\Phi'(-h_i)]^2}\frac{\Phi(\rho)}{\rho+h_i}+\frac{1}{(1-q)[\Phi'(-1)]^2}\frac{\Phi(\rho)}{\rho+1},$$

$$g=\frac{3}{4q}\sum_{i=1}^n \frac{1-h_i}{h_i[\Phi'(-h_i)]^2}, \qquad \mathscr{g}=\frac{1}{4q}\sum_{i=1}^n \frac{1}{(h_i-q)[\Phi'(-h_i)]^2}+\frac{1}{2q(1-q)[\Phi'(-1)]^2},$$

for $m=2n+1$.

Then

$$T_m=\left(1+g\frac{q}{\rho}\mathsf{E}^2\right)R+\mathscr{g}\,\frac{q(1-q)^2}{(\rho+1)(\rho+q)}\,\mathsf{E}^2Q-K\Delta,$$

$$K=\left[P+g\frac{\mathsf{E}}{\rho}-\mathscr{g}\left(\frac{q}{\rho+1}+\frac{1}{\rho+q}\right)\mathsf{E}\right]\mathsf{E}\qquad\text{for } m=2n,$$

$$=\left[P+g\frac{\Phi}{\rho}-\mathscr{g}\left(\frac{q}{\rho+1}+\frac{1}{\rho+q}\right)\Phi\right]\frac{\mathsf{E}}{\sqrt{\rho+1}}\qquad\text{for } m=2n+1;$$

$T_m=0$ is considered a function of two independent variables ρ and q. Then we take ρ as the only independent variable by considering $T_m=0$ together with $T_{23}=0$.

Next, Liapounov computed

$$A_2=\frac{1}{\mathsf{E}}\left(\mathsf{J}\frac{d}{d\rho}\frac{\mathsf{E}^2\mathsf{F}}{\Delta}-\frac{\mathsf{E}^2\mathsf{F}}{\Delta}\frac{d\mathsf{J}}{d\rho}\right),$$

$$\mathsf{J}=\frac{1}{(2m+1)\gamma}\int\frac{[E(\mu)E(\nu)]^3}{(\rho+\mu^2)(\rho+\nu^2)}\,d\sigma,$$

for $m=2n$ and obtained A_2 in a form which is rational in ρ and algebraic in q; $A_2\neq 0$ for any value of ρ or q. It is shown that $\Omega>0$ for the values of ρ, q such as $T_{23}=T_m=0$, and that $A_2>0$ for $T_{23}=T_m$. He used almost 100 pages to prove $A_3>0$ for $m=3$.

Let E_0 be the singular ellipsoid, E_η be a Jacobi ellipsoid slightly different from E_0 with singular velocity $\Omega_0+\eta$, and F be a nonellipsoidal figure with $\Omega_0+\eta$. The differences of the moments of inertia $S-S_\eta$, $S-S_0$, and the differences of the moments of momentum $M-M_\eta$, $M-M_0$ are negative in the order of η. Hence M and S decrease in the transition to a pear-shaped figure. Thus he concluded that a pear-shaped figure is unstable.

NEW FORMULAS

In Part IV, Liapounov presented a new formula, which is the starting point of his discussion on heterogeneous masses. This was published in two volumes after his death.

At first he took

$$x=\sqrt{1+\zeta}\,\sqrt{\rho+1}\,\sin\theta\cos\psi,\qquad y=\sqrt{1+\zeta}\,\sqrt{\rho+q}\,\sin\theta\sin\psi,\qquad z=\sqrt{1+\zeta}\,\sqrt{\rho}\,\cos\theta,$$

but later took the equations of a new equilibrium figure to be

$$\left.\begin{aligned}x&=a\sqrt{1+\bar\zeta}\,\sqrt{\rho+1}\,\sin\theta\cos\psi+\beta\sqrt{\rho+1},\\ y&=a\sqrt{1+\bar\zeta}\,\sqrt{\rho+q}\,\sin\theta\sin\psi,\\ z&=a\sqrt{1+\bar\zeta}\,\sqrt{\rho}\,\cos\theta,\quad 0\leqslant a\leqslant 1,\end{aligned}\right\}\qquad(208)$$

where β is a function of α that reduces to 0 with α. He expanded

$$\bar\zeta=\sum\bar\zeta_{rs}\alpha^r\beta^s,\qquad \bar\zeta_{r0}=\zeta_r;$$

note that $a=1$ corresponds to the free surface, and $0<a<1$ corresponds to one of the level surfaces that are similar and similarly situated with the free surface. A level surface is defined by $U+\Omega(x^2+y^2)=$ function of a, and is represented by

$$x'=\sqrt{u}\,\sqrt{\rho+1}\,\sin\theta'\cos\psi'+\beta\sqrt{\rho+1}$$
$$y'=\sqrt{u}\,\sqrt{\rho+q}\,\sin\theta'\sin\psi',$$
$$z'=\sqrt{u}\,\sqrt{\rho}\,\cos\theta',$$

where u is a function of θ', ψ' such that $0<u<1+\bar{\zeta}'$.

Put

$$U_0=\frac{\Delta}{2\pi}\int d\sigma'\int_0^1\frac{\sqrt{u}\,du}{D(a,\sqrt{u})},\qquad S=\frac{1}{4\pi}\int d\sigma'\int_{1+\zeta}^{1+\bar{\zeta}'}\frac{\sqrt{u}\,du}{D(a\sqrt{1+\zeta},\sqrt{u})},$$
$$\Delta=\sqrt{\rho(\rho+1)(\rho+q)},\qquad r=D(a\sqrt{1+\zeta},\sqrt{u}),$$

then

$$U=(1+\zeta)U_0+2\Delta S.$$

Put, further,

$$\Omega-\Omega_0=\eta,\qquad \Theta=(\rho+\cos^2\psi+q\sin^2\psi)\sin^2\theta,\qquad \frac{1}{2}\int_\rho^\infty\frac{dt}{\Delta(t)}=C,\qquad \frac{\rho}{2}\int_\rho^\infty\frac{dt}{t\Delta(t)}=R;$$

then,

$$R\zeta=\frac{\eta}{2\Delta}(1+\zeta)\Theta+\frac{(t+1)(\Omega_0+q)}{\Delta a}\beta\sqrt{1+\zeta}\sin\theta\cos\psi+\frac{1}{a^2}(C\zeta+S)+\frac{f(a)}{a^2},\tag{209}$$

where $f(a)$ is an indeterminate function of a. The problem is to determine the function ζ, satisfying this equation by a suitable choice of β and $f(a)$. In order that ζ shall not become infinite for $a=0$, we should have

$$f(0)=-\frac{1}{4\pi}\int\frac{\bar{\zeta}'d\sigma'}{D(0,1)},$$
$$\frac{\Omega_0+\eta}{\Delta}\beta+\frac{1}{2\pi}\int\frac{\sin\theta'\cos\psi'}{D^3(0,1)}(\sqrt{1+\bar{\zeta}'}-1)\,d\sigma'=0;$$

$$\frac{f(a)-f(0)}{a^2}\quad\text{is finite for}\quad a=0.$$

The second condition fixes β as a positive integral power series of α, vanishing for $\alpha=0$. The first condition gives $f(0)$; $f(a)=f(1)a^2+f(0)(1-a^2)$ satisfies the third condition. Substitute these expressions for $\bar{\zeta}'$, $f(a)$, β, η into equation 209; then equation 209 is expressed in terms of ζ, a, θ, ψ, and α. The solution ζ of equation 209 tends to zero for $\alpha=0$, whatever the value of a between 0 and 1.

Next, expand

$$S=\sum_{n=1}^{\infty}S_n;$$

$S_n (n \geqslant 2)$ is the coefficient of ϵ^n in the expansion of

$$\frac{1}{4\pi}\int d\sigma' \int_1^{1+\epsilon\bar{\zeta}'} \frac{\sqrt{u}\,du}{D(a\sqrt{1+\epsilon\zeta},\ \sqrt{u})}.$$

Expand $\bar{\zeta}'$ in powers of α such that

$$\bar{\zeta}' = \sum_{n=1} J_n \alpha^n,$$

the coefficients J_n being functions of $v = a\sqrt{1+\epsilon\zeta}$. Then equation 209 is written

$$R\zeta = f(1) - f(0) + \frac{\eta}{2\Delta}(1+\zeta)\Theta + \frac{1}{a^2}[J(a\sqrt{1+\zeta}) - J(0) - J'(0)a\sqrt{1+\zeta}]. \tag{210}$$

Since the right-hand side is developed in positive integral powers of α and ζ and becomes zero for $\alpha = 0$, equation 210 admits a unique solution

$$\zeta = \sum_{r=1} \zeta_r \alpha^r$$

for a sufficiently small α. Compute the coefficients successively, starting at ζ_1. Put $f(1) - f(0) = c$

$$J(v) - J(0) - J'(0)v + \left(\frac{\eta}{2\Delta}\Theta + c\right)v^2 = F(v), \qquad \xi = \frac{F(a)}{(R+c)a^2};$$

then

$$\zeta = \frac{1}{a^2}\sum_{n=1}^{\infty} \frac{1}{n!}\frac{\partial^{n-1} a^{2n}\xi^n}{\partial (a^2)^{n-1}}; \tag{211}$$

function ξ is developed in powers of α and is a known function of a, θ, ψ, α; and ζ_r is obtained as an integral polynomial of a, $\sin\theta\cos\psi$, $\cos\theta$. Inserting this definition of ξ in equation 210, we obtain

$$\zeta = (1+\zeta)\xi(a\sqrt{1+\zeta}).$$

Putting

$$a\sqrt{1+\zeta} = v, \qquad v^2[1 - \xi(v)] = a^2$$

gives

$$v^2 - \Phi(v\sin\theta\cos\psi, v\sin\theta\sin\psi, v\cos\theta) = a^2,$$

where Φ is a uniform, analytic function of the arguments, whose development begins with the second-degree terms in the arguments. Hence, we obtain finally from equation 208 the equation for an interior level surface of the equilibrium figure in the form

$$\frac{(x-\lambda)^2}{\rho+1} + \frac{y^2}{\rho+q} + \frac{z^2}{\rho} - \Phi\left(\frac{x-\lambda}{\sqrt{\rho+1}}, \frac{y}{\sqrt{\rho+q}}, \frac{z}{\sqrt{\rho}}\right) = a^2, \tag{212}$$

with $\beta\sqrt{\rho+1} = \lambda$. The lowest-degree terms in the development of Φ are of the form

$$L\frac{(x-\lambda)^2}{\rho+1}+M\frac{y^2}{\rho+q}+N\frac{z^2}{\rho^2},$$

where L, M, N become zero for $\alpha=0$. Or, if we neglect the terms of order higher than a^2, the equation for the surface (equation 212) is

$$(1-L)\frac{(x-\lambda)^2}{\rho+1}+(1-M)\frac{y^2}{\rho+q}+(1-N)\frac{z^2}{\rho}=a^2.$$

This is an ellipsoid with its center at $(\lambda, 0, 0)$; $\lambda=0$ for m even; the ellipsoid is concentric with E_0. If m is odd, if E_0 is a figure of revolution, and if $k(\text{of } T_{m,2k}=0)>1$, then $\lambda=0$. Substituting our expansion $\lambda=\lambda_1\alpha+\lambda_2\alpha^2+\ldots$, we obtain

$$\frac{x^2}{\rho+1}+\frac{y^2}{\rho+q}+\frac{z^2}{\rho}=a^2+\sum_{i=1}^{\infty}\alpha^i Z_i(x, z, a).$$

Put $a=1$ for the free surface; then we obtain the equation that was given formerly:

$$\frac{x^2}{\rho+1}+\frac{y^2}{\rho+q}+\frac{z^2}{\rho}=1+\sum_{i=1}^{\infty}\alpha^i Z_i(x, z).$$

Liapounov wrote $\alpha^2=\int\zeta^2 d\sigma$, expanded all quantities in powers of α, and determined ζ in the form

$$\zeta=\sum\zeta_{rs}\alpha^r\eta^s.$$

Finally, he expressed η as a power series of α and then obtained the solution ζ in a convergent power series of α.

In 1917, Liapounov solved the problem by means of spherical functions with the supposition that the level surfaces are homothetic. But if we consider them to be confocal, then Lamé functions appear (Liapounov, 1903, 1904, 1925, 1927).

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CHAPTER VII

Theory of Lichtenstein

NONLINEAR INTEGRAL EQUATIONS

Integral Power Series

According to Schmidt (1908), Iglisch (1933), Hammerstein (1930), and Golomb (1934), we call an expression

$$u(s)^{\alpha_0}\int\int \ldots \int K(s, t_1, \ldots, t_\rho)u(t_1)^{\alpha_1}u(t_2)^{\alpha_2} \ldots u(t_\rho)^{\alpha_\rho}dt_1 \ldots dt_\rho,$$

$$\alpha_0+\alpha_1 + \ldots +\alpha_\rho=m, \qquad \alpha_0 \geqslant 0, \qquad \alpha_1 \geqslant 1, \qquad \ldots \alpha_\rho \geqslant 1,$$

an *integral power term* of degree m in the *argument function* $u(s)$, where the *coefficient function* K is continuous in $a \leqslant s \leqslant b$, $a \leqslant t_1 \leqslant b, \ldots, a \leqslant t_\rho \leqslant b$, and $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_\rho$. The product of an integral power term of degree m and an integral power term of degree n is an integral power term of degree $m+n$.

If we substitute for the argument function $u(s)$ of an integral power term of degree n an integral power term of degree m in an argument function $v(s)$, then we obtain an integral power term of degree mn in the argument function $v(s)$.

Denote the integral power terms of degree m in an argument function $u(s)$ by $W_m\binom{s}{u}$, $V_m\binom{s}{u}$, $P_m\binom{s}{u}$; then we have $W_m\binom{s}{pu}=p^mW_m\binom{s}{u}$ with a constant p, and $W_m\binom{s}{p}=p^mW_m\binom{s}{1}$ with $u(s)=1$.

Let an integral power term, in which the coefficient function is replaced by its absolute value, be denoted by $|W|_m\binom{s}{u}$; then we see that $W_0(s)+W_1\binom{s}{u}+ \ldots +W_m\binom{s}{u}+ \ldots$ is a regular convergent integral power series $\mathfrak{P}\binom{s}{u}$ which represents a continuous function of s if the series $|\tilde{W}|_0+|\tilde{W}|_1\tilde{u}+|\tilde{W}|_2\tilde{u}^2+ \ldots$ is convergent, where the maximum of the absolute value of $u(s)$, $W_m\binom{s}{1}$, $|W|_m\binom{s}{1}$ is denoted respectively by $\tilde{u}$, $\tilde{W}_m$, $|\tilde{W}|_m$. If the argument function is replaced by an argument function smaller than the maximum of the first argument function in its absolute value, then the series thus obtained is absolutely and uniformly convergent. The product and the sum of two absolutely and uniformly convergent integral power series is an absolutely and uniformly convergent integral power series.

We extend this to the case of two-argument functions. Consider

$$W_{mn}\binom{s}{uv} \equiv u(s)^{\alpha_0}v(s)^{\beta_0}\int \ldots \int K(s, t_1, t_2, \ldots, t_p)u(t_1)^{\alpha_1}v_1(t_1)^{\beta_1} \ldots u(t_p)^{\alpha_p}v(t_p)^{\beta_p}dt_1 \ldots dt_p,$$

$$\alpha_0+\alpha_1+, \ldots +\alpha_p=m, \qquad \beta_0+ \ldots +\beta_p=n,$$
$$\alpha_1+\beta_1 \geqslant 1, \qquad \ldots \alpha_p+\beta_p \geqslant 1,$$
$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_p,$$

where

$$\beta_\mu \geqslant \beta_\nu \quad \text{if} \quad \alpha_\mu=\alpha_\nu.$$

The number of integral power terms of degree m in $u(s)$ and of degree n in $v(s)$ is finite. The product of an integral power term of degree m in $u(s)$ and of degree n in $v(s)$ with an integral power term of degree m' in $u(s)$ and of degree n' in $v(s)$ is an integral power term of degree $m+m'$ in $u(s)$ and of degree $n+n'$ in $v(s)$.

With constants p, q, we have

$$W_{mn}\binom{s}{pu,\ qv}=p^mq^nW_{mn}\binom{s}{uv}, \qquad W_{mn}\binom{s}{pq}=p^mq^n\binom{s}{11}.$$

If

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|W|_{mn}\tilde{u}^m\tilde{v}^n$$

is convergent, then the integral power series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W_{mn}\binom{s}{uv}$$

is also convergent.

If $W(s)=W_0(s)+W_1\binom{s}{u}+ \ldots$ and $u(s)=V_1\binom{s}{v}+V_2\binom{s}{v}+ \ldots$ are both absolutely and uniformly convergent such that $\tilde{u} \leqslant h$, $|V|_1\tilde{v}+|\tilde{V}|_2\tilde{v}^2+ \ldots \leqslant h$, then the integral power series in the argument function $v(s)$ obtained by substituting such $u(s)$ in $W_i\binom{s}{u}$ is also absolutely and uniformly convergent.

Consider an absolutely and uniformly convergent integral power series $H(S)=\mathfrak{P}\binom{s}{uv}$ for $\tilde{u} \leqslant h$, $\tilde{v} \leqslant k$, where

$$u(s)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}W_{mnp}\binom{s}{w_1w_2w_3}, \tag{213}$$

$$v(s)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}V_{mnp}\binom{s}{w_1w_2w_3}, \tag{214}$$

$$W_{000}(s)\equiv V_{000}(s)=0.$$

The integral power series obtained by substituting these $u(s)$, $v(s)$ in $H(S)$ is absolutely and uniformly convergent if $u(s)$, $v(s)$ are absolutely and uniformly convergent and if

$$\left.\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}|\tilde{W}|_{mnp}\tilde{w}_1^m\tilde{w}_2^n\tilde{w}_3^p\leq h,\\&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}|\tilde{V}|_{mnp}\tilde{w}_1^m\tilde{w}_2^n\tilde{w}_3^p\leq h.\end{aligned}\right\} \tag{215}$$

(Cf., Niemytzki, 1933; Bratu, 1913).

Inversion

Now consider the inversion of the integral power series

$$\mathfrak{P}\binom{s}{uv}\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W_{mn}\binom{s}{uv}=0, \tag{216}$$

in which the series $\mathfrak{P}\binom{s}{uv}$ is absolutely and uniformly convergent for $\tilde{u}<h$, $\tilde{v}<k$, and $W_{00}(s)=0$, so that $\mathfrak{P}\binom{s}{00}=0$. Try to express $u(s)$ of equation 216 as a function of $v(s)$. An integral power term of degree 1 in $u(s)$ and of degree 0 in $v(s)$ is

$$\left.\begin{aligned}&W_{10}\binom{s}{uv}=A(s)u(s)+\int_a^b B(s,t)u(t)dt,\\ &\text{or}\\ &W_{10}\binom{s}{uv}=u(s)+\int_a^b C(s,t)u(t)dt.\end{aligned}\right\} \tag{217}$$

There are two cases: (1) when there is no continuous function $\varphi(s)$ satisfying

$$\varphi(s)-\int C(s,t)\varphi(t)dt=0, \tag{218}$$

that is, for the kernel $C(s, t)$ there is no null-solution (which means that there is no such solution that is not identically zero and satisfies the homogeneous integral equation); and (2) when there are a certain number of such solutions.

The Regular Case

In the regular case when there is no null-solution of equation 218, there is one and only one solution $u(s)$ for a given function $v(s)$, if the continuous functions $u(s)$ and $v(s)$ satisfy

$u \leqslant h'$, $\tilde{v} \leqslant k'$ with $h' \leqslant h$, $k' \leqslant k$. Then $u(s)$ can be written in the form of an absolutely and uniformly convergent integral power series with $v(s)$ as the argument function.

Proof: Equation 216 with equation 217 can be written

$$u(s) - \int C(s, t)u(t)dt = -W_{01}\binom{s}{uv} - \sum_{m+n\geqslant 2} W_{mn}\binom{s}{uv}. \tag{219}$$

From the theory of Fredholm we can obtain the resolvent $\Gamma(s, t)$ such that

$$f(s) + \int \Gamma(s, t)f(t)dt = \varphi(s)$$

of the nonhomogeneous integral equation

$$\varphi(s) - \int C(s, t)\varphi(t)dt = f(s).$$

Equation 219 is equivalent to

$$u(s) - W_{01}\binom{s}{uv} - \int \Gamma(s, t)W_{01}\binom{t}{uv}dt + \sum_{m+n\geqslant 2}\left[-W_{mn}\binom{s}{uv} - \int \Gamma(s, t)W_{mn}\binom{t}{uv}dt\right]. \tag{220}$$

Write

$$-W_{mn}\binom{s}{uv} - \int \Gamma(s, t)W_{mn}\binom{t}{uv}dt \equiv P_{mn}\binom{s}{uv}$$

and

$$P_{01}\binom{s}{uv} \equiv P_1\binom{s}{v};$$

then,

$$u(s) = P_1\binom{s}{v} + \sum_{m+n\geqslant 2} P_{mn}\binom{s}{uv} \tag{221}$$

is absolutely and uniformly convergent for $\tilde{u} < h$, $\tilde{v} < k$.

Write $V_1\binom{s}{v} = P_1\binom{s}{v}$ and solve equation 221 for any value of m by putting

$$\sum_{\nu-1}^{m-1} V_\nu\binom{s}{v} \quad \text{for} \quad u(s).$$

We then have

$$u(s) = \sum_{m=1}^{\infty} V_m\binom{s}{v} \tag{222}$$

as a formal solution of equation 216.

It can be shown that there exists k_1 such that

$$\sum_{m=1}^{\infty} |\tilde{V}|_m \tilde{v}^m$$

is convergent; that is, the series of equation 222 is absolutely and uniformly convergent for $\tilde{v} \leqslant k_1$. Hence there exists such a positive number $h' \leqslant h$, that equation 216 has one and only one solution for $\tilde{u} \leqslant h'$, $\tilde{v} \leqslant h'$.

The Bifurcation Case

In the bifurcation case, when there are a certain number n of the null-solutions of equation 218, consider the associated equation

$$\psi(s) - \int C(t, s)\psi(t)\,dt = 0 \tag{223}$$

to equation 218. Equations 218 and 223 each have n linearly independent solutions

$$\varphi_1(s),\ \varphi_2(s),\ \ldots\ \varphi_n(s),$$

$$\psi_1(s),\ \psi_2(s),\ \ldots\ \psi_n(s),$$

and the general solutions are respectively

$$\sum_{\nu=1}^{n} c_\nu \varphi_\nu(s), \qquad \sum_{\nu=1}^{n} c_\nu \psi_\nu(t),$$

with arbitrary constants $c_1, c_2, \ldots, c_n$. With real or complex functions $p(s)$ and $q(t)$, we form

$$E(s, t) = C(s, t) + \sum_{\nu=1}^{n} p_\nu(s) q_\nu(t), \tag{224}$$

and put

$$A_{\mu\nu} = \int \psi_\mu(r) p_\nu(r)\,dr, \qquad B_{\mu\nu} = \int \varphi_\mu(r) q_\nu(r)\,dr. \tag{225}$$

The necessary and sufficient condition for $E(s, t)$ to have no null-solution, is that the determinants $\|A_{\mu\nu}\|$, $\|B_{\mu\nu}\|$ are not zero. Equation 224 is sometimes called the *kernel-transformation.*

Suppose at first that equations 218 and 223 each have only one null-solution. Let them be $c_1\varphi_1(s)$ and $c_1\psi_1(t)$. Form

$$E(s, t) = C(s, t) + p_1(s) q_1(t) \tag{226}$$

with $p_1(s)$ and $q_1(t)$ such that

$$\int \psi_1(r) p_1(r)\,dr \neq 0, \qquad \int \varphi_1(r) q_1(r)\,dr \neq 0. \tag{227}$$

This new kernel $E(s, t)$ then has no null-solution. Denote the resolvent by $\mathfrak{E}(s, t)$; then from equation 219, we have

$$u(s) - \int E(st) u(t)\,dt = -p_1(s) \int q_1(t) u(t)\,dt - W_{01}\binom{s}{uv} - \sum_{m+n\geqslant 2} W_{mn}\binom{s}{uv}. \tag{228}$$

Put

$$-W_{mn}\binom{s}{uv}-\int \mathfrak{G}(s,\,t)\,W_{mn}\binom{t}{uv}dt=P_{mn}\binom{s}{uv}, \qquad P_{01}\binom{s}{uv}=P_1\binom{s}{v};$$

then equation 228 can be written

$$u(s)=\left[-p_1(s)-\int \mathfrak{G}(s,\,t)p_1(t)dt\right]\int q_1(t)u(t)dt+P_1\binom{s}{v}+\sum_{m+n\geqslant 2} P_{mn}\binom{s}{uv},$$

or

$$u(s)=\left[-p_1(s)\int \mathfrak{G}(s,\,t)p_1(t)dt\right]x+P_1\binom{s}{v}+\sum_{m+n\geqslant 2} P_{mn}\binom{s}{uv}, \tag{229}$$

with

$$x=\int q_1(t)u(t)dt. \tag{230}$$

We can solve equation 229 for $u(s)$ with x as a parameter, and the result is

$$u(s)=\sum_{m+n\geqslant 1} x^m V_n^m\binom{s}{v} \tag{231}$$

if $\tilde{v}\leqslant k_1\leqslant k$, $\tilde{u}\leqslant h_1\leqslant h$, $|x|\leqslant \ell_1$, $(\ell_1>0)$, where $V_n^m\binom{s}{v}$ is an integral power series of degree n in $v(s)$. Function $u(s)$, considered as an integral power series of x and $v(s)$, is absolutely and uniformly convergent. Substituting this series for $u(t)$ in equation 230, we obtain

$$x=\sum_{m+n\geqslant 1} x^m\int q_1(t)V_n^m\binom{t}{v}dt.$$

Also, write

$$\int q_1(t)V_0^m\binom{t}{v}dt=L_m \qquad (m=1,\,2,\,.\,.\,.); \tag{232}$$

then, since $L_1=1$, we obtain

$$0=\sum_{m=2}^{\infty} L_m x^m+\sum_{m=0}^{\infty} x^m\cdot\sum_{n=1}^{\infty}\int V_n^m\binom{t}{v}q_1(t)dt. \tag{233}$$

This is called the *bifurcation equation.* If $v(s)$ is given, we can obtain the solution of equation 216 by substituting each root of equation 233 such as $|x|\leqslant \ell_1$ in equation 231, for $\tilde{v}\leqslant k_1$, $\tilde{u}\leqslant h_1$, $|x|\leqslant \ell_1$.

Suppose that $L_2\neq 0$ in equation 233 and put

$$S_1=\sum_{m=2}^{\infty} L_m x^m, \qquad S_2=\sum_{m=0}^{\infty} x^m\sum_{n=1}^{\infty}\int V_n^m\binom{t}{v}q_1(t)dt.$$

Fix a positive number $\ell_2\leqslant \ell_1$ such that $0\leqslant |x|\leqslant \ell_2$. We can choose a nonzero positive number $k_2\leqslant k_1$ so that $|S_2|\leqslant \alpha\sigma_1$ with a pure fraction α for $|x|=\ell_2$, $\tilde{v}\leqslant k_2$, where σ_1

denotes the minimum of $|S_1|$. The number of solutions of equation 233 that do not exceed ℓ_2 is given by Kronecker's theorem

$$\frac{1}{2\pi i}\int_{|x|=\ell_2}\frac{\dfrac{\partial S_1}{\partial x}+\dfrac{\partial S_2}{\partial x}}{S_1+S_2}\,dx,$$

an integer, which is two in our case. Thus we obtain two solutions of equation 216 for $L_2\neq 0$. Hence equation 216 has a double bifurcation point at $u(s)=0$, $v(s)=0$.

Next suppose that $L_2=L_3=\ldots=L_{n-1}=0$, $L_n\neq 0$. Then an n-ple bifurcation takes place at $u(s)=0$, $v(s)=0$. The question is whether the solutions are all real. Let equation 233 be of the form

$$0=L_nx^n+A_0v(s)+A_{00}v^2(s)+A_{01}v(s)x+\ldots,$$

all the coefficients being real; then there are only two solutions if n is even and $A_0L_n<0$, and there is only one solution for n odd. If equation 233 is of the form $A_{00}v^2+A_{01}vx+L_2x^2+\ldots=0$, then there are two, one, or no solutions according as

$$A_{01}^2-4L_2A_{00}>0,\ =0,\ \text{or} <0.$$

Next suppose that equations 218 and 223 have two null-solutions, $p_1(s)$, $p_2(s)$ and $q_1(t)$, $q_2(t)$, and that

$$\begin{vmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix}\neq 0,\qquad \begin{vmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{vmatrix}\neq 0. \tag{234}$$

Then form a new kernel $E(s, t)$ by the so-called kernel-transformation

$$E(s,t)=C(s,t)+p_1(s)q_1(t)+p_2(s)q_2(t).$$

Function $E(s, t)$ has no null-solution. Denote the resolvent by $\mathfrak{E}(s, t)$, and put

$$x=\int q_1(t)u(t)dt,\qquad y=\int q_2(t)u(t)dt. \tag{235}$$

From

$$u(s)=\left[-p_1(s)-\int\mathfrak{E}(s,t)p_1(t)dt\right]x$$
$$+\left[-p_2(s)-\int\mathfrak{E}(s,t)p_2(t)dt\right]y+P_1\binom{s}{v}+\sum_{m+n\geq 2}P_{mn}\binom{s}{uv} \tag{236}$$

we obtain an absolutely and uniformly convergent integral power series for $v(s)$, x, y:

$$u(s)=\sum_{\alpha+\beta+n\geq 1}x^\alpha y^\beta V_n^{\alpha\beta}\binom{s}{v}, \tag{237}$$

where $V_n^{\alpha\beta}\binom{s}{v}$ is an integral power series for $v(s)$, if $\tilde v\leq k_1$, $\tilde u\leq h_1$, $|x|\leq\ell_1$, $|y|\leq\ell_1'$ with suitably chosen positive numbers ℓ_1, ℓ_1', $k_1\leq k$, $h_1\leq h$. Substituting equation 237 in equation 235, we obtain

$$\left.\begin{aligned} x &= \sum_{\alpha+\beta\geq 1} L_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\alpha+\beta\geq 0} x^{\alpha}y^{\beta}\sum_{n=1}^{\infty}\int V_n^{\alpha\beta}\binom{t}{v}q_1(t)\,dt, \\ y &= \sum_{\alpha+\beta\geq 1} L'_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\alpha+\beta\geq 0} x^{\alpha}y^{\beta}\sum_{n=1}^{\infty}\int V_n^{\alpha\beta}\binom{t}{v}q_2(t)\,dt, \end{aligned}\right\} \tag{238}$$

where

$$L_{\alpha\beta} = \int V_0^{\alpha\beta}\binom{t}{v}q_1(t)\,dt, \quad L'_{\alpha\beta} = \int V_0^{\alpha\beta}\binom{t}{v}q_2(t)\,dt. \tag{239}$$

The sums on the right-hand sides are an absolutely and uniformly convergent integral power series for $x, y, v(s)$ and vanish with $v(s)$.

Now, since $\varphi_1(t)$ and $\varphi_2(t)$ are linearly independent solutions of equation 218, we obtain from equation 236

$$\begin{aligned} \varphi_1(s) = \left[-p_1(s) - \int \mathfrak{G}(s,t)p_1(t)\,dt\right] \int q_1(t)\varphi_1(t)\,dt \\ + \left[-p_2(s) - \int \mathfrak{G}(s,t)p_2(t)\,dt\right] \int q_2(t)\varphi_1(t)\,dt, \\ \varphi_2(s) = \left[-p_1(s) - \int \mathfrak{G}(s,t)p_1(t)\,dt\right] \int q_1(t)\varphi_2(t)\,dt \\ + \left[-p_2(s) - \int \mathfrak{G}(s,t)p_2(t)\,dt\right] \int q_2(t)\varphi_2(t)\,dt; \end{aligned}$$

or

$$\left.\begin{aligned} \varphi_1(s) &= V_0^{10}(s)B_{11} + V_0^{01}(s)B_{12}, \\ \varphi_2(s) &= V_0^{10}(s)B_{21} + V_0^{10}(s)B_{22}, \end{aligned}\right\} \tag{240}$$

where

$$B_{\mu\nu} = \int \varphi_\mu(r)q_\nu(r)\,dr, \qquad \mu, \nu = 1, 2.$$

Intergrating these equations after multiplying at first by $q_1(s)ds$ and next by $q_2(s)ds$, and referring to equation 239, we obtain

$$\left.\begin{aligned} 0 &= B_{11}(L_{10}-1) + B_{12}L_{01} \\ 0 &= B_{21}(L_{10}-1) + B_{22}L_{01} \end{aligned}\right\} \qquad \left.\begin{aligned} 0 &= B_{11}L'_{10} + B_{12}(L'_{01}-1), \\ 0 &= B_{21}L'_{10} + B_{22}(L'_{01}-1). \end{aligned}\right\}$$

We see from equation 234 that $L_{10}=1$, $L_{01}=0$, $L'_{10}=0$, $L'_{01}=1$. Hence equation 238 can be written

$$\left.\begin{aligned} 0 &= \sum_{\alpha+\beta\geq 2} L_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\alpha+\beta\geq 0} x^{\alpha}y^{\beta}\sum_{n=1}^{\infty}\int V_n^{\alpha\beta}\binom{t}{v}q_1(t)\,dt, \\ 0 &= \sum_{\alpha+\beta\geq 2} L'_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\alpha+\beta\geq 0} x^{\alpha}y^{\beta}\sum_{n=1}^{\infty}\int V_n^{\alpha\beta}\binom{t}{v}q_2(t)\,dt. \end{aligned}\right\} \tag{241}$$

These are the *equations of bifurcation.* (See Iglisch, 1929, 1930, 1930a, 1931, 1933). Lichtenstein (1931) deduced similar results by a different method. It has been generalized to higher dimensions (Levi, 1907). This discussion of bifurcation can be applied to differential equations in Sturm-Liouville's problem (Falckenberg, 1912).

Nonlinear Integro-Differential Equation

Put

$$Du=\frac{du(s)}{ds}, \qquad Du(t_i)=\frac{d}{ds}u(t_i),$$

and consider with

$$U_{mnp}\{u, Du, v\} \equiv \sum_j u(s)^{\alpha_0} v(s)^{\gamma_0} \int \ldots \int K_{mnpj}(s, t_1, \ldots, t_\rho)$$
$$\times v(t_1)^{\gamma^1} \ldots u(t_\rho)^{\alpha_\rho} [Du(t_1)]^{\beta_1} \ldots [Du(t_\rho)]^{\beta_\rho}$$
$$\times v(t_1)^{\gamma_1} \ldots v(t_\rho)^{\gamma_\rho} dt_1 \ldots dt_\rho,$$
$$\alpha_0+\alpha_1+\ldots+\alpha_\rho=m, \qquad \beta_1+\beta_2+\ldots+\beta_\rho=n,$$
$$\gamma_0+\gamma_1+\ldots+\gamma_\rho=p, \qquad \rho=m+n+p,$$

an integro-differential equation

$$\sum_{m+n+p>0} U_{mnp}\{u, Du, v\}=0, \tag{242}$$

or

$$u(s)+\int K_{1002}(s, t_1)u(t_1)dt_1+\int K_{0101}(s, t_1)Du(t_1)dt_1$$
$$=L(s)v(s)-\int K_{0012}(s, t_1)v(t_1)dt_1-\sum_{m+n+p\geq 2} U_{mnp}\{u, Du, v\}.$$

We have the bifurcation or the regular case according as the integral equation

$$u(s)+\int K_{1002}(s, t_1)u(t_1)dt_1=0$$

has null-solutions or not.

Lichtenstein first considered simultaneous nonlinear integral equations

$$U^{(1)}_{mnp}\{u, v, w\}=0, \qquad U^{(2)}_{mnp}\{u, v, w\}=0,$$

or simultaneous linear integral equations

$$\Lambda_1\{u, v\} \equiv u(s)+\int_0^1 L_1^{(1)}(s, t_1)u(t_1)dt_1+\int_0^1 L_2^{(1)}(s, t_1)v(t_1)dt_1=f(s),$$
$$\Lambda_2\{u, v\} \equiv v(s)+\int_0^1 L_1^{(2)}(s, t_1)u(t_1)dt_1+\int_0 L_2^{(2)}(s, t_1)v(t_1)dt_1=g(s).$$

After reducing them to a single, linear integral equation as suggested by a remark of Fredholm, he discussed the regular and the bifurcation cases for this linear, integral equation and then returned to the n simultaneous nonlinear integro-differential equations.

EQUILIBRIUM FIGURES

Fundamental Integro-Differential Equation

Let the coordinates of a point on the surface S bounding the body T (which may consist of several components) of an equilibrium figure be denoted by $x=X(\xi, \eta)$, $y=Y(\xi, \eta)$, $z=Z(\xi, \eta)$ such that

$$\left[\frac{\partial(X, Y)}{\partial(\xi, \eta)}\right]^2+\left[\frac{\partial(Y, Z)}{\partial(\xi, \eta)}\right]^2+\left[\frac{\partial(Z, X)}{\partial(\xi, \eta)}\right]^2 \neq 0.$$

Denote the Newtonian potential of the body T by $V(x, y, z)$; then

$$V(X, Y, Z)+\frac{\omega^2}{2\kappa f}(X^2+Y^2)$$

is constant on each component of S, where ω is the rotation velocity such that $\omega < \sqrt{2\pi\kappa f}$, κ is the Gaussian constant, and f is the density supposed to be constant. The resultant of the attracting force and the centrifugal force has been shown to be directed inward or zero, and $z=0$ and $y=0$ are two symmetry planes whose existence has already been proved. Suppose that if we keep the volume constant there exists an equilibrium figure T_1 with ω_1 in the neighborhood of T with ω. Then we should have

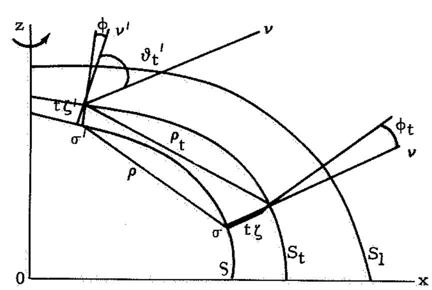
$$V_1(X_1, Y_1, Z_1)-V(X, Y, Z)=\frac{\omega^2}{2\kappa f}(X^2+Y^2)-\frac{\omega_1^2}{2\kappa f}(X_1^2+Y_1^2)+s, \tag{243}$$

where s is a fixed constant on each component. If T consists of only one component, then $s=0$. Draw an outward normal ν at a point $(\xi\eta)$ on S and take ζ on that normal; then S_1 is represented by $\zeta=\zeta(\xi, \eta, \omega_1)$. Let the distance of (ξ, η) from the z-axis be R, and the cosine of the angle between ν and the perpendicular from (ξ, η) to the z-axis be τ. Furthermore, let (ξ, η, ζ^*) be the coordinates of a point referred to the xyz-axis system, so that $V(x, y, z)=W(\xi, \eta, \zeta^*)$, $V_1(x, y, z)=W_1(\xi, \eta, \zeta^*)$, and let the distance between a point (ξ, η) and a point (ξ', η') be ρ. The attracting force due to T on the point (ξ, η) of mass 1 in the ν-direction is $f\kappa(\partial/\partial\nu)W(\xi, \eta, 0)$, and the gravity there is normal to S and equal to

$$f\kappa\frac{\partial}{\partial\nu}W(\xi, \eta, 0)+\omega^2R\tau=f\kappa\psi, \tag{244}$$

where $\psi<0$ since the gravity is directed inward. Write $V_1(X_1, Y_1, Z_1)=U_1(\xi, \eta)$, or $V(X, Y, Z)=U(\xi, \eta)$ when the point (ξ, η) is on the surface S_1 or S, respectively.

We consider a one-parametric family of surfaces $S_t (0\leq t\leq 1)$, between S and S_1, and denote it by $S+t(S_1-S)$. A point $(\xi, \eta, t\zeta)$ on S_t corresponds to a point (ξ, η) on S. Denote the potential due to T_t (which is the body enclosed by S_t) at the point (ξ, η, ζ^*) by $W_t(\xi, \eta, \zeta^*)$, and let the potential at $(\xi, \eta, t\zeta)$ be $U_t(\xi, \eta)$. Furthermore, let φ_t be the angle between ν and the outward normal at $(\xi, \eta, t\zeta)$ *to* S_t, $d\sigma_t'$ be the surface element of S_t at $(\xi', \eta', t\zeta')$, and ϑ_t' be the angle between ν and the outward normal to S_t at $(\xi', \eta', t\zeta')$. The equation for S is $x=X(\xi, \eta)$, $y=Y(\xi, \eta)$, $z=Z(\xi, \eta)$, and the equation for S_t is $x=X+at\zeta$, $y=Y+bt\zeta$, $z=Z+ct\zeta$, where a, b, c are the direction cosines of ν. Thus



$$\left.\begin{aligned}
&d\sigma_t'=d\xi'd\eta'\sqrt{A_t'^2+B_t'^2+C_t'^2},\\
&A_t'=\frac{\partial(Y'+b't\zeta',\,Z'+c't\zeta')}{\partial(\xi',\,\eta')},\qquad B_t'=\frac{\partial(Z'+c't\zeta',\,X'+a't\zeta')}{\partial(\xi',\,\eta')},\\
&C_t'=\frac{\partial(X'+a't\zeta',\,Y'+b't\zeta')}{\partial(\xi',\,\eta')},\qquad \cos\varphi_t'=\frac{a'A_t'+b'B_t+c'C_t'}{\sqrt{A_t'^2+B_t'^2+C_t'^2}},\\
&\cos\vartheta_t'=\frac{aA_t'+bB_t'+cC_t'}{\sqrt{A_t'^2+B_t'^2+C_t'^2}}.
\end{aligned}\right\} \tag{245}$$

It can be proved that U_1-U is uniformly convergent for sufficiently small

$$|\zeta|,\ \left|\frac{\partial\zeta}{\partial\xi}\right|,\ \left|\frac{\partial\zeta}{\partial\eta}\right|,$$

so that

$$U_1-U=U^{(1)}+U^{(2)}+\ .\ .\ .$$

$$U^{(n)}=\int_S\frac{1}{\rho^n}K^{(n)}d\sigma',$$

$$\frac{1}{\rho^n}K^{(n)}=\frac{1}{n\,!}\frac{1}{\sqrt{A'^2+B'^2+C'^2}}\sum(a'\zeta'-a\zeta)\left[A_t'\frac{\partial^{n-1}}{\partial t^{n-1}}\left(\frac{1}{\rho_t}\right)+\binom{n-1}{1}\frac{\partial A_t'}{\partial t}\frac{\partial^{n-2}}{\partial t^{n-2}}\left(\frac{1}{\rho_t}\right)\right.$$
$$\left.+\binom{n-1}{2}\frac{\partial^2A_t'}{\partial t^2}\frac{\partial^{n-3}}{\partial t^{n-3}}\left(\frac{1}{\rho_t}\right)\right]t=0,$$

where

$$A'=\frac{\partial(Y',Z')}{\partial(\xi',\eta')},\qquad B'=\frac{\partial(Z',X')}{\partial(\xi',\eta')},\qquad C'=\frac{\partial(X',Y')}{\partial(\xi',\eta')}. \tag{246}$$

In particular, for $n=1$, we have

$$U^{(1)}=\zeta\frac{\partial}{\partial\nu}W(\xi,\eta,0)+\int_S\frac{1}{\rho}\zeta'd\sigma'. \tag{247}$$

It is proved that

$$|U^{(2)}+U^{(3)}+\ .\ .\ .\ |\equiv|\Psi|,$$

and also

$$\left|\frac{\partial\Psi}{\partial\xi}\right|,\ \left|\frac{\partial\Psi}{\partial\eta}\right|,\ \left|\frac{\partial^2\Psi}{\partial\xi^2}\right|,\ \left|\frac{\partial^2\Psi}{\partial\xi\partial\eta}\right|,\ \left|\frac{\partial^2\Psi}{\partial\eta^2}\right|$$

are all bounded by $<\gamma\epsilon^2$ with a finite positive constant γ, if

$$\left|\frac{\partial^2\zeta}{\partial\xi^2}\right|,\ \left|\frac{\partial^2\zeta}{\partial\xi\partial\eta}\right|,\ \left|\frac{\partial^2\zeta}{\partial\eta^2}\right|<\epsilon.$$

Write

$$\frac{\omega_1^2-\omega^2}{2\kappa f}=\lambda$$

and note that $X_1^2+Y_1^2=R^2+2R\tau\zeta+(a^2+b^2)\zeta^2$. Then, from equations 249, 247, 243, and 244, we obtain our fundamental nonlinear integro-differential equation for determining ζ:

$$\psi\zeta+\int_S \frac{1}{\rho}\zeta' d\sigma' = s-R^2\lambda-\frac{\omega^2}{2\kappa f}(a^2+b^2)\zeta^2-2R\tau\lambda\zeta-(a^2+b^2)\lambda\zeta^2-U^{(2)}-U^{(3)}. \tag{248}$$

This equation is of a form similar to that of Liapounov.

Existence Theorem

The linear homogeneous integral equation

$$\psi\zeta+\int_S \frac{1}{\rho}\zeta' d\sigma'=0 \qquad (\psi<0) \tag{249}$$

has at least two linearly independent null-solutions if S is not a surface of revolution around the z-axis. If S is a surface of revolution around the z-axis, then there exists always only one null-solution. The proof is given by displacing S by a small amount parallel to the z-axis, and then rotating S through a small angle around the z-axis. Such null-solutions u_1 and u_2 satisfy

$$\int_S \psi\, u_1^2 d\sigma=-1, \qquad \int_S \psi\, u_2^2 d\sigma=-1. \tag{250}$$

If we displace a nearby equilibrium figure S_1 by a small amount parallel to the z-axis, or if we rotate it through a small angle, then we obtain the same equilibrium figure. Hence we put

$$\int_S \psi\, u_1\zeta d\sigma=0, \tag{251}$$

$$\int_S \psi\, u_2\zeta d\sigma=0, \tag{252}$$

in order to fix the position of S_1. When S is a surface of revolution, then $u_2=0$, and equation 252 is automatically satisfied. Put

$$\frac{1}{4\rho}=\psi'(u_1u_1'+u_2u_2')+\frac{1}{\psi}N \tag{253}$$

and let $y=0$, $z=0$ be the symmetry plane of S. Let σ^*, $\sigma^{*\prime}$ be the image of the point σ, σ' with regard to $y=0$; then $\rho(\sigma^*, \sigma^{*\prime})=\rho(\sigma, \sigma')$, $u_1(\sigma^*)=u_1(\sigma)$, $u_2(\sigma^*)=-u_2(\sigma)$, $\psi(\sigma^*)=\psi(\sigma)$. Hence $N(\sigma^*, \sigma^{*\prime})=N(\sigma, \sigma')$. Let σ_*, σ_*' be the image of the point σ, σ' with regard to $z=0$; then $N(\sigma_*, \sigma_*')=N(\sigma, \sigma')$. Referring to equations 251, 252, and 253, we transform our fundamental equation 248 into

$$\begin{aligned}\psi\zeta+\int_S N\zeta' d\sigma' &= s-R^2\lambda-\frac{\omega^2}{2\kappa f}(a^2+b^2)\zeta^2-2R\tau\lambda\zeta\\ &\quad-(a^2+b^2)\lambda\zeta^2-U^{(2)}-U^{(3)}\ \ldots\\ &=\Pi\{\lambda, s, \zeta\}.\end{aligned} \tag{254}$$

Our next problem is to determine whether equation 249 admits any null-solution other than u_1 and u_2.

The Regular Case

Suppose that there is no null-solution, besides u_1 and u_2, of equation 249. By the well-known theorem, no null-solution exists for

$$\psi\zeta+\int_S N\zeta' d\sigma'=0. \tag{255}$$

Making successive approximations, we obtain the solution for equation 254 by solving the following set of equations for $\zeta_1, \zeta_2, \ldots$:

$$\psi\zeta_1+\int_S N\zeta_1' d\sigma'=s-R^2\lambda,$$

$$\psi\zeta_2+\int_S N\zeta_2' d\sigma'=\Pi\{\lambda, s, \zeta_1\},$$

$$\psi\zeta_3+\int_S N\zeta_3' d\sigma'=\Pi\{\lambda, s, \zeta_2\}, \;\ldots\;.$$

It can be proved that the series

$$\sum_{n=1}^{\infty} \zeta_n$$

is convergent insofar as $|\lambda|$, $|s|$ are sufficiently small.

The Bifurcation Case

Suppose that there are $m-2$ null-solutions $u_3, u_4, \ldots, u_m$ besides u_1, u_2 of equation 249; $u_1, u_2, u_3, \ldots u_m$ are the linearly independent, complete system of solutions for equation 249. Since $y=0$, $z=0$ are the symmetry planes, $v_3(\sigma)=u_3(\sigma^*), \ldots, v_m(\sigma)=u_m(\sigma^*)$ are null solutions of equation 249; $w_i=u_i+v_i$, $\bar{w}_i=u_i-v_i$, $(i=1, 2, \ldots, m)$, are also null solutions. If we denote a linearly independent, complete system of null-solutions by $w_{1i}(\sigma)$, $(i=1, 2, \ldots, m)$, then either $w_{1i}(\sigma^*)=w_{1i}(\sigma)$ or $w_{1i}(\sigma^*)=-w_{1i}(\sigma)$. If S is not a surface of revolution, then $u_2(\sigma^*)=-u_2(\sigma)$, and hence $w_1=2u_1$, $\bar{w}_2=2u_2$; thus u_1 and u_2 are included in this complete system w_{1i}. Similarly we repeat the argument for the symmetry plane $z=0$ and find that u_1, u_2 are included in the new complete system. Hence we consider u_1, u_2, $u_3 \ldots, u_m$ to be our linearly independent complete system of null-solutions of equation 249. Let u_i be the linearly independent, complete system of null-solutions, such that

$$\psi u_i+\int_S \frac{1}{\rho} u_i' d\sigma'=0,$$

with

$$\int_S \psi u_i^2 d\sigma=-1 \qquad (i=1, 2, \ldots, m).$$

Suppose that $u_3, \ldots, u_m$ are symmetric with regard to $y=0$ and $z=0$. Then

$$\psi N-\sum_{i=3}^{m}\psi' u_i u_i' =\frac{1}{\psi\rho}-\sum_{i=1}^{m}\psi' u_i u_i' =\frac{1}{\psi}N_1$$

satisfies $N_1(\sigma^*, \sigma^{*\prime})=N_1(\sigma, \sigma')$, $N_1(\sigma_*, \sigma_*')=N_1(\sigma, \sigma')$, and the integral equation

$$\psi\zeta+\int_S N_1\zeta' d\sigma'=0$$

does not have any null-solution. Write equation 254 in the form

$$\psi\zeta+\int_S N_1\zeta' d\sigma'=\Pi\{\lambda, s, \zeta\}+\sum_{i=3}^{m}\psi r_i u_i, \tag{256}$$

$$r_i=-\int_S \psi' u_i' \zeta' d\sigma' \qquad (i=3, 4, \ldots m), \tag{257}$$

and consider r_i at first to be indeterminate parameters. If $|\lambda|$, $|s|$, and $|r_i|$ $(i=3, 4, \ldots, m)$, are sufficiently small, then it can be shown that there exists one and only one solution ζ which is analytic with regard to these m parameters λ, s, r_i.

Let H_1 be the resolvent for the kernel $(1/\psi)N_1$. Since

$$\int_S H_1(\sigma, \sigma')u_i(\sigma')d\sigma'=0,$$

we obtain

$$\begin{aligned}\zeta=\frac{s}{\psi}+\sum_{j=3}^{m} r_j u_j-\frac{1}{\psi}R^2\lambda+\int_S H_1\left(-\frac{s}{\psi}+\frac{1}{\psi}R'^2\lambda\right)d\sigma'&\\ +\sum f_{\nu_0\nu_1\mu_3\ldots\mu_m}\lambda^{\nu_0}s^{\nu_1}r_3^{\mu_3}\ldots r_m^{\mu_m} \qquad (\nu_0+\nu_1+\mu_3+\ldots+\mu_m>1).&\end{aligned} \tag{258}$$

Since ζ is symmetric with respect to $y=0$ and $z=0$, we have

$$\int_S \zeta u_1 d\sigma=0, \qquad \int \zeta u_2 d\sigma=0.$$

Substituting equation 258 in equation 257, we obtain $m-2$ equations for $r_3, \ldots, r_m$. Since

$$\int_S \psi u_i^2 d\sigma=-1, \qquad \text{and} \qquad \int_S H_1(\sigma, \sigma')\psi(\sigma)u_i(\sigma)d\sigma=0 \qquad (i=3, \ldots m),$$

from the theory of linear integral equations, the result is

$$\left.\begin{aligned}&\lambda\int R^2 u_i d\sigma-s\int_S u_i d\sigma+\mathfrak{P}_i(\lambda, s; r_3, \ldots, r_m)=0,\\ &\mathfrak{P}_i=\sum B_{\nu_0\nu_1\mu_3\ldots\mu_m}\lambda^{\nu_0}s^{\nu_1}r_3^{\mu_3}\ldots r_m^{\mu_m}\\ &(\nu_0+\nu_1+\mu_3+\ldots+\mu_m>1).\end{aligned}\right\} \tag{259}$$

We thus obtain the solution $r_j = r_j(\lambda, s)$ $(j=3, \ldots, m)$ which vanishes for $\lambda = s = 0$. Hence the study of equilibrium figures in the neighborhood of the given figure is reduced to the discussion of equation 259.

We divide a linear series into two parts $\lambda \geqslant 0$ and $\lambda \leqslant 0$, and call each part an *arm*. If we can develop in convergent power series of λ in the neighborhood of $\lambda=0$, then we call the linear series *regular*. A regular series consists of two arms $\lambda > 0$ and $\lambda < 0$.

For simplicity, put $s=0$, $m=3$; then u_3 is the only new null-solution. Equation 259 is written

$$\mathsf{A}\lambda + \mathfrak{P}(\lambda, r_3) = 0,$$

$$\mathsf{A} = \int_S R^2 u_3 d\sigma, \qquad \mathfrak{P}(\lambda, r_3) = \sum_{j,l} B_{jl}\lambda^j r_3^l \qquad (j+l>1). \tag{260}$$

Let $\mathsf{A} \neq 0$ and $B_{0k} r_3^k$ $(k \geqq 2)$ be the first nonvanishing coefficient. Then, near $\lambda=0$, we have

$$r_3 = \left(-\frac{\mathsf{A}}{B_{0k}}\lambda\right)^{1/k} + \mathfrak{P}^{(1)}\left[\left(1-\frac{\mathsf{A}}{B_{0k}}\lambda\right)^{1/k}\right],$$

where $\mathfrak{P}^{(1)}$ is a power series in λ which vanishes for $\lambda=0$. If k is even, there are two real series of equilibrium figures, and two arms, for which $-(\mathsf{A}/B_{0k})\lambda > 0$. If k is odd, then there is only one arm for $\lambda > 0$ and one for $\lambda < 0$.

Next suppose that $\mathsf{A}=0$, $B_{02} \neq 0$; then equation 260 is written explicitly $B_{02}r_3^2 + B_{11}r_3\lambda + B_{20}\lambda^2 + \ldots = 0$. If $B_{11}^2 - 4B_{02}B_{20} > 0$, we have two regular series crossing each other. If $B_{11}^2 - 4B_{20} < 0$, then there is no real equilibrium figure in the neighborhood, and S is an isolated figure. If $B_{11}^2 - 4B_{02}B_{20} = 0$, then we must study the higher-degree terms.

Suppose that $\mathsf{A}=0$, $B_{02}=0$, $B_{11} \neq 0$, $B_{20} \neq 0$. We have

$$B_{11}r_3\lambda + B_{0n}r_3^n + B_{0n_1}r_3^{n_1} + \ldots + \lambda^2(B_{20} + \ldots) + \ldots = 0 \qquad (n>2).$$

In this case we have one regular series

$$r_3 = -\frac{B_{20}}{B_{11}}\lambda + \mathfrak{P}(\lambda)$$

and two real arms

$$r_3 = \pm\left(-\frac{B_{11}}{B_{0n}}\lambda\right)^{1/(n-1)}$$

for n odd, $B_{11}/B_{0n} < 0$, and one real arm

$$r_3 = \left(-\frac{B_{11}}{B_{0n}}\lambda\right)^{1/(n-1)}$$

for n even (Lichtenstein, 1917, 1920; Schur, 1919. Cf., Lichtenstein, 1921, 1922, 1922a, 1927, 1928; Hölder, 1926, 1929, 1933; Kähler, 1928; Garten, 1932.) The equilibrium figures of a heterogeneous liquid mass have also been studied after the fashion of Liapounov (Lense, 1923; Lichtenstein, 1923a, 1933, 1933a; Hölder, 1933; Maruhn, 1933, 1934).

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Boundary Value Problems

Consider a boundary value problem which leads to a nonlinear integro-differential equation

$$\frac{d}{dx}\left(p\,\frac{d\zeta}{dx}\right)+q\zeta+\int \mu(x, x_1)\zeta(x_1)dx_1=U(x)$$
$$-\sum_{m+n>1}\ \sum_{j}\int\ .\ .\ .\ \int L_{mnj}(x; x_1, \ldots, x_\rho)\zeta^{\alpha}\zeta_1^{\alpha_1}\ldots\zeta_\rho^{\alpha_\rho}v^{\beta}v_1^{\beta_1}\ldots v_\rho^{\beta_\rho}dx_1\ldots dx_\rho \tag{261}$$

with the boundary condition

$$\zeta(0)=\zeta(1)=0, \tag{262}$$

where $p(x)$, $q(x)$ are continuous, $M(x, x_1)$ is continuous and symmetric, and the series on the right-hand side is absolutely and uniformly convergent as long as $|\zeta|$, $|v|$ are sufficiently small.

A homogeneous differential equation

$$\Lambda(\zeta)\equiv\frac{d}{dx}\left(p\,\frac{d\zeta}{dx}\right)+q\zeta=0 \tag{263}$$

cannot have more than one solution that vanishes at 0 and 1 and is continuous with its first and second-order derivatives. If such a solution does not exist, for example in the case $q<0$, then there exists Green's function $G(\xi, x)=G(x, \xi)$ such that it is continuous in $0\leqslant x\leqslant 1$, $0\leqslant\xi\leqslant 1$ and satisfies equation 263 as a function of x in $0<\xi<1$, $x\neq\xi$, and also satisfies equation 262, i.e.,

$$\Lambda[G(\xi, x)]=0, \qquad G(\xi, 0)=G(\xi, 1)=0;$$

its first-order derivative jumps at $x=\xi$, such that

$$\frac{\partial}{\partial x}G(\xi, \xi+0)-\frac{\partial}{\partial x}G(\xi, \xi-0)=-\frac{1}{p(\xi)}\cdot$$

In this case, a nonhomogeneous equation

$$\frac{d}{dx}\left(p\,\frac{d\zeta}{dx}\right)+q\zeta=h(x) \tag{264}$$

has one and only one solution

$$\zeta(\xi)=-\int G(\xi, x)h(x)dx. \tag{265}$$

If the homogeneous equation (equation 53) has a solution $u(x)$, such as

$$\int_0^1 [u(x)]^2dx=1,$$

then we have Green's function with an extended sense such that

$$\Lambda[\mathfrak{G}(\xi, x)] = u(x)u(\xi), \qquad \mathfrak{G}(\xi, 0) = \mathfrak{G}(\xi, 1) = 0, \qquad \int \mathfrak{G}(\xi, x)u(x)dx = 0,$$

$$\frac{\partial}{\partial x}\mathfrak{G}(\xi, \xi+0) - \frac{\partial}{\partial x}\mathfrak{G}(\xi, \xi-0) = -\frac{1}{p(\xi)}, \qquad \mathfrak{G}(\xi, x) = \mathfrak{G}(x, \xi).$$

Then nonhomogeneous equation 264 is soluble only when

$$\int h(x)u(x)dx = 0, \tag{266}$$

and the solution is

$$\zeta(\xi) = -\int \mathfrak{G}(\xi, x)h(x)dx + cu(\xi), \qquad c = \text{arbitrary} \tag{267}$$

(Hilbert, 1912).

When equation 263 has no solution, we obtain, putting

$$\int G(x, x')M(x', x_1)dx = -N(x, x_1),$$

$$G(x, x')L_{mnj}(x', x_1, \ldots, x_\rho) = -N_{mnj}(x; x', x_1, \ldots, x_\rho),$$

a nonlinear integral equation

$$\zeta(x) + \int N(x, x_1)\zeta(x_1)dx_1 = -\int G(x, x')U(x')dx'$$

$$-\sum_{m+n>1}\sum_j \int \cdots \int N_{mnj}(x; x', x_1, \ldots, x_\rho)\zeta^\alpha(x')\zeta_1^{\alpha_1} \ldots \zeta_\rho^{\alpha_\rho} v^\beta v_1^{\beta_1} \ldots v_\rho^{\beta_\rho} dx' dx_1 \ldots dx_p.$$

According as the homogeneous equation

$$\zeta(x) + \int N(x, x_1)\zeta(x_1)dx_1 = 0$$

has a null-solution or not, there occurs the bifurcation or the regular case. If the linear integro-differential equation

$$\frac{d}{dx}\left(p\frac{d\zeta}{dx}\right) + q\zeta + \int M(x, x_1)\zeta(x_1)dx_1 = 0 \tag{268}$$

has no solution vanishing at 0 and 1, the regular case occurs. If the equation has null-solutions, then the bifurcation occurs.

When equation 263 has a null-solution $u(x)$, then equation 261 is written

$$\frac{d}{dx}\left(p\frac{d\zeta}{dx}\right) + q_1\zeta = (q_1 - q)\zeta - \int M(x, x_1)\zeta(x_1)dx_1 + U(x)$$

$$-\sum_{m+n>1}\sum_j \int \cdots \int L_{mnj}(x; x_1, \ldots, x_\rho)\zeta^\alpha\zeta_1^{\alpha_1} \ldots \zeta_\rho^{\alpha_\rho} v^\beta v_1^{\beta_1} \ldots v_\rho^{\beta_\rho} dx_1 \ldots dx_\rho,$$

$$(q_1 < 0 \text{ constant}).$$

Let the Green function for

$$\frac{d}{dx}\left(p\frac{d\zeta}{dx}\right)+q_1\zeta=0$$

be $\hat{G}(\xi, x)$; then,

$$\zeta(x)+\int \hat{G}(x,x')[q_1-q(x')]\zeta(x')dx'-\int\int \hat{G}(x,x')M(x',x_1)\zeta(x_1)dx_1dx'$$
$$=-\int \hat{G}(x,x')U(x')dx'+\sum_{m+n>1}\sum_j\int\ldots\int \hat{G}(x,x')L_{mnj}(x';x_1,\ldots,x_\rho)$$
$$\times\zeta^\alpha(x')\zeta_1^{\alpha_1}\ldots v^\beta(x')v_1^{\beta_1}\ldots dx'dx_1\ldots dx_\rho.$$

According as equation 268 has a null-solution or not, there occurs the bifurcation and the regular case (Lichtenstein, 1931).

Saturn's Ring

Lichtenstein applied the theory to the oscillation of the incoherent particles forming the rings of Saturn (1923, 1924, 1932, 1933). Consider a mass M at the coordinate origin and a constant density distribution of μ along the circle $C: x+y^2=R^2$. The ring rotates with a uniform angular velocity ω around a fixed coordinate system xyz. The position of a particle P_0 of the ring is defined by R and s/R; we count s along the ring. A disturbing mass $\mathfrak{M}$ is supposed to be at distance $\mathfrak{R}$ from the center of the ring, and particle P_0 is disturbed to a position $P[R+\zeta, (s+\sigma)/R]$; while ζ and σ are functions of s/R and t. Assume that $|\zeta|<R$; then

$$x=(R+\zeta)\cos\left(\omega t+\frac{s+\sigma}{R}\right),\qquad y=(R+\zeta)\sin\left(\omega t+\frac{s+\sigma}{R}\right).$$

Since the kinetic energy of the particle is

$$T=(1/2)\left[\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2\right]=(1/2)(R+\zeta)^2\left(\omega+\frac{1}{R}\frac{\partial\sigma}{\partial t}\right)^2+(1/2)\left(\frac{\partial\zeta}{\partial t}\right)^2,$$

the Lagrangian equations of motion are

$$\frac{\partial^2\zeta}{\partial t^2}-(R+\zeta)\left(\omega+\frac{1}{R}\frac{d\sigma}{dt}\right)^2=Q_\zeta,\qquad \frac{1}{R^2}(R+\zeta)^2\frac{\partial^2\sigma}{\partial t^2}+2(R+\zeta)\left(\omega+\frac{1}{R}\frac{\partial\sigma}{\partial t}\right)\frac{1}{R}\frac{\partial\zeta}{\partial t}=Q_\sigma,$$

where $Q_\zeta\delta\zeta$, $Q_\sigma\delta\sigma$ are virtual work due to gravitation. Let ν be the direction of the normal to C at P and τ be the direction perpendicular to ν. Then, for the attraction of the ring to the particle at P, we have

$$Q_\zeta=\kappa\int_C\mu' ds'\frac{\partial}{\partial\nu}\log\frac{R}{\rho},\qquad Q_\sigma=\kappa\left(1+\frac{\zeta}{R}\right)\int_C\mu' ds'\frac{\partial}{\partial\tau}\log\frac{R}{\rho}.$$

Let the distance between the disturbing mass $\mathfrak{M}$ and P be P; then for this disturbed motion, we have

$$Q_\zeta = \kappa\mathfrak{M}\frac{\partial}{\partial\nu}\log\frac{R}{\mathrm{P}}, \qquad Q_\sigma = \kappa\mathfrak{M}\left(1+\frac{\zeta}{R}\right)\frac{\partial}{\partial\tau}\log\frac{R}{\mathrm{P}}.$$

For the attraction of the central body, we have

$$Q_\zeta = -\frac{M\kappa}{R+\zeta}, \qquad Q_\sigma = 0.$$

Hence the Lagrangian equations of our problem are

$$\left.\begin{aligned}
&\frac{\partial^2\zeta}{\partial t^2} - (R-\zeta)\left(\omega+\frac{1}{R}\frac{\partial\sigma}{\partial t}\right)^2 = -\frac{M\kappa}{R+\zeta} + \kappa\int_C \mu' ds' \frac{\partial}{\partial\nu}\log\frac{R}{\rho} + \kappa\mathfrak{M}\frac{\partial}{\partial\nu}\log\frac{R}{\mathrm{P}},\\
&(R+\zeta)^2\frac{1}{R^2}\frac{\partial^2\sigma}{\partial t^2} + 2(R+\zeta)\left(\omega+\frac{1}{R}\frac{\partial\sigma}{\partial t}\right)\frac{1}{R}\frac{\partial\zeta}{\partial t}\\
&\qquad = \kappa\left(1+\frac{\zeta}{R}\right)\int_C \mu' ds' \frac{\partial}{\partial\tau}\log\frac{R}{\rho} + \kappa\left(1+\frac{\zeta}{R}\right)\mathfrak{M}\frac{\partial}{\partial\nu}\log\frac{R}{\mathrm{P}}.
\end{aligned}\right\} \tag{269}$$

Denote the distance of two points $(R, s/R)$ and $(R, s'/R)$ on C by ρ_0, the outward normal to C by ν_0, and the tangent by τ_0; then we have

$$\int_C \mu' ds' \frac{\partial}{\partial\nu_0}\log\frac{R}{\rho_0} = -\pi\mu, \qquad \int_C \mu' ds' \frac{\partial}{\partial\tau_0}\log\frac{R}{\rho_0} = 0.$$

Also, we have

$$\omega^2 = \frac{\kappa}{R^2}\left(M+\pi R\mu\right).$$

Put

$$\nu = \omega^2 + \frac{M\kappa}{R^2} = 2\omega^2 - \frac{\kappa\pi\mu}{R}.$$

Let β be the angular velocity of $\mathfrak{M}$, and $\mathfrak{R}$, ϑ be the polar coordinates of P with reference to the fixed coordinates x, y. Then $\vartheta = \vartheta_0 + \beta t$;

$$\mathrm{P}^2 = (R+\zeta)^2 + \mathfrak{R}^2 - 2\mathfrak{R}(R+\zeta)\cos\left(\omega t + \frac{s+\sigma}{R}\right),$$

$$\rho^2 = (R+\zeta)^2 + (R+\zeta')^2 - 2(R+\zeta)(R+\zeta')\cos\left(\frac{s+\sigma}{R} - \frac{s'+\sigma'}{R}\right).$$

Write $\beta - \omega = \gamma$.

Suppose that the solution of equation 269 is of the form

$$\left.\begin{aligned}
\zeta &= \zeta(s, t) = Z(s-R\gamma t) = Z(u),\\
\sigma &= \sigma(s, t) = S(s-R\gamma t) = S(u),
\end{aligned}\right\} \tag{270}$$

where $Z(u)$, $S(u)$ are periodic functions of period $2\pi R$. We have

$$\frac{\partial \zeta}{\partial t} = -R\gamma \frac{dZ}{du}, \qquad \frac{\partial^2 \zeta}{\partial t^2} = R^2\gamma^2 \frac{d^2Z}{du^2},$$

$$\frac{\partial \sigma}{\partial t} = -R\gamma \frac{dS}{du}, \qquad \frac{\partial^2 \sigma}{\partial t^2} = R^2\gamma^2 \frac{d^2S}{du^2}.$$

Hence our differential equations become

$$\left.\begin{aligned} L_1(Z, S) &\equiv R^2\gamma^2 \frac{d^2Z}{du^2} + 2R\gamma\omega \frac{dS}{du} - \nu Z \\ &= R\gamma^2 \left(\frac{dS}{du}\right)^2 - 2\gamma\omega Z \frac{dS}{du} + \gamma^2 Z \left(\frac{dS}{du}\right)^2 - \frac{M\kappa}{R}\left(\frac{Z^2}{R^2} - \frac{Z^3}{R^3} + \cdots\right) \\ &\quad + \kappa \int_C \mu' du' \left(\frac{\partial}{\partial \nu} \log \frac{R}{\rho} - \frac{\partial}{\partial \nu_0} \log \frac{R}{\rho_0}\right) + \kappa \mathfrak{M} \frac{\partial}{\partial v} \log \frac{R}{\mathsf{P}} \equiv \Lambda_1(Z, S), \\ L_2(Z, S) &\equiv R^2\gamma^2 \frac{d^2S}{du^2} - 2\gamma R\omega \frac{dZ}{du} \\ &= -\gamma^2 RZ \frac{d^2S}{du^2} - 2\gamma^2 R \frac{dS}{du}\frac{dZ}{du} \\ &\quad + \kappa \int_C \mu' du' \left(\frac{\partial}{\partial \tau} \log \frac{R}{\rho} - \frac{\partial}{\partial \tau_0} \log \frac{R}{\rho_0}\right) + \mu \mathfrak{M} \frac{\partial}{\partial \tau} \log \frac{R}{\mathsf{P}} \equiv \Lambda_2(Z, S) \end{aligned}\right\} \quad (271)$$

At first we consider

$$\left.\begin{aligned} R^2\gamma^2 \frac{d^2Z}{du^2} + 2R\gamma\omega \frac{dS}{du} - \nu Z &= F_1(u), \\ R^2\gamma^2 \frac{d^2S}{du^2} - 2R\gamma\omega \frac{dZ}{du} &= F_2(u), \end{aligned}\right\} \quad (272)$$

where F_1, F_2 are periodic with period $2\pi R$. It is questioned whether periodic solutions exist with the same period as F_1 and F_2.

The solution of the homogeneous equations from equation 272 is

$$\left.\begin{aligned} Z(u) &= B + C \cos \frac{pu}{\gamma R} + D \sin \frac{pu}{\gamma R}, \\ S(u) &= A + B \frac{\nu u}{2R\gamma\omega} + Cq \sin \frac{pu}{\gamma R} - Dq \cos \frac{pu}{\gamma R}, \end{aligned}\right\} \quad (273)$$

where

$$p = \sqrt{2\omega^2 + \frac{\kappa\pi\mu}{R}}, \qquad q = \frac{2\omega}{\sqrt{2\omega^2 + \dfrac{\kappa\pi\mu}{R}}}.$$

If we set $\sqrt{2\omega^2 + \kappa\pi\mu/R} = \gamma$, then the homogeneous equations admit three linearly independent periodic solutions:

$$0,\ R; \qquad R\cos\frac{u}{R},\ \frac{2R\omega}{\gamma}\sin\frac{u}{R}; \qquad R\sin\frac{u}{R},\ -\frac{2R\omega}{\gamma}\cos\frac{u}{R}. \tag{274}$$

Suppose at first that $\gamma^2 \neq 2\omega^2+\kappa\pi\mu/R$; then the first solution $Z(u)=0$, $S(u)=R$ is the only periodic solution with period $2\pi R$. Let $\mathfrak{G}(\xi, u)$ be the periodic Green function for

$$\frac{d^2Z}{du^2}-\frac{\nu}{\gamma^2R^2}Z=0,$$

such that

$$\mathfrak{G}(\xi, 0)=\mathfrak{G}(\xi, 2\pi R), \qquad \frac{\partial}{\partial u}\mathfrak{G}(\xi, 0)=\frac{\partial}{\partial u}\mathfrak{G}(\xi, 2\pi R).$$

Then the solution of equation 273 is written

$$\left.\begin{aligned} Z(\xi) &= \frac{2\omega}{\gamma R}\int_0^{\pi R}\mathfrak{G}(\xi, u)\frac{dS}{du}\,du-\frac{1}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)F_1(u)\,du,\\ S(\xi) &= -\frac{2\omega}{\gamma R}\int_0^{2\pi R}\mathfrak{G}(\xi, u)\frac{dZ}{du}\,du+\frac{\nu}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)S(u)\,du\\ &\quad -\frac{1}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)F_2(u)\,du, \end{aligned}\right\}$$

or, integrating by parts,

$$\left.\begin{aligned} Z(\xi) &= -\frac{2\omega}{\gamma R}\int_0^{2\pi R}\frac{\partial}{\partial u}\mathfrak{G}(\xi, u)S(u)\,du-\frac{1}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)F_1(u)\,du,\\ S(\xi) &= \frac{2\omega}{\gamma R}\int_0^{2\pi R}\frac{\partial}{\partial u}\mathfrak{G}(\xi, u)Z(u)\,du+\frac{\nu}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)S(u)\,du\\ &\quad -\frac{1}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)F_2(u)\,du. \end{aligned}\right\} \tag{275}$$

Evidently the homogeneous integral equations from equation 275 admit the solution $Z=0$, $S=R$. Hence, putting $Z=0$, $S=R$, $F_1=F_2=0$ in equation 275, we obtain

$$\int_0^{2\pi R}\frac{\partial}{\partial u}\mathfrak{G}(\xi, u)\,du=0, \qquad \frac{\nu}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi, u)\,du=\frac{\nu}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(u, \xi)\,du=1, \tag{276}$$

and hence

$$\int_0^{2\pi R}\frac{\partial}{\partial \xi}\mathfrak{G}(\xi, u)\,du=0.$$

Let the integration domain E be two circuits of $(0, 2\pi R)$; put $z(\xi)=Z(\xi)$ for the first circuit and $z(\xi)=S(\xi)$ for the second. Put $\Phi(\xi)=\Phi^{(1)}(\xi)$ in the first circuit and $\Phi(\xi)=\Phi^{(2)}(\xi)$ in the second circuit, where

$$\Phi^{(1)}(\xi)=-\frac{1}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi,u)F_1(u)du,\qquad \Phi^{(2)}(\xi)=-\frac{1}{\gamma^2R^2}\int_0^{2\pi R}\mathfrak{G}(\xi,u)F_2(u)du;$$

let $K(\xi, u)$ be as shown in Table II.

TABLE II.—*Integration Domain E Circuits*

	1st circuit of u	2nd circuit of u
1st circuit of ξ:	0	$-\frac{2\omega}{\gamma R}\frac{\partial}{\partial u}\mathfrak{G}(\xi,u)$
2nd circuit of ξ:	$\frac{2\omega}{\gamma R}\frac{\partial}{\partial u}\mathfrak{G}(\xi,u)$	$\frac{\nu}{\gamma^2R^2}\mathfrak{G}(\xi,u)$

then equation 275 is written as a single integral equation

$$Z(\xi)=\int_E K(\xi,u)Z(u)du+\Phi(\xi). \tag{277}$$

The corresponding homogeneous integral equation has a null-solution $W(\xi)$, which is equal to 0 in the first circuit and equal to R in the second circuit. The necessary and sufficient condition for the solubility of equation 277 is

$$-\frac{1}{\gamma^2R^2}\int_0^{2\pi R}d\xi\int_0^{2\pi R}\mathfrak{G}(\xi,u)F_2(u)du=0,$$

or, by equation 276,

$$\int_0^{2\pi R}F_2(u)du=0. \tag{278}$$

With the resolvent $H(\xi, u)$ the solution is

$$Z(\xi)=\int_E H(\xi,u)\Phi(u)du+cW(s), \tag{279}$$

where c is a constant; c is seen to be

$$c=\frac{1}{2\pi R^2}\int_0^{2\pi R}S(\xi)d\xi.$$

Put $c=0$. Then

$$|S|,\quad |Z|,\quad \left|\frac{dS}{du}\right|,\quad \left|\frac{dZ}{du}\right|,\quad \left|\frac{d^2S}{du^2}\right|,\quad \left|\frac{d^2Z}{du^2}\right|<h\qquad \text{if } |F_1(u)|, |F_2(u)|<h,$$

especially, if $F_1(u)$, $F_2(u)$ are respectively symmetric and antisymmetric with regard to the direction $0\mathfrak{M}$, that is, if

$$F_1(u)=F_1(2\vartheta_0-u),\ \text{ or }\ F_2(u)=-F_2(2\vartheta_0-u); \tag{280}$$

then the solution for $c=0$ is

$$Z(u)=Z(2\vartheta_0-u), \text{ or } S(u)=-S(2\vartheta_0-u). \tag{281}$$

We apply this theorem to our nonlinear integro-differential equations (equation 271) and write

$$F_1(u)=R\gamma^2\left(\frac{dS}{du}\right)^2-2\gamma\omega Z\frac{dS}{du}+\gamma^2 Z\left(\frac{dS}{du}\right)^2-\frac{\mu\kappa}{R}\left(\frac{Z^2}{R^2}-\frac{Z^3}{R^3}-\ldots\right)$$
$$+\kappa\int_C\mu' du'\left(\frac{\partial}{\partial\nu}\log\frac{R}{\rho}-\frac{\partial}{\partial\nu_0}\log\frac{R}{\rho_0}\right)+\kappa\mathfrak{M}\frac{\partial}{\partial\nu}\log\frac{R}{\mathrm{P}};$$

$$F_2(u)=-\gamma^2 RZ\frac{d^2S}{du^2}-2\gamma^2 R\frac{dS}{du}\frac{dZ}{du}+\kappa\int_C\mu' du'\left(\frac{\partial}{\partial\tau}\log\frac{R}{\rho}-\frac{\partial}{\gamma\tau_0}\log\frac{R}{\rho_0}\right)+\kappa\mathfrak{M}\frac{\partial}{\partial\nu}\log\frac{R}{\mathrm{P}}.$$

The integrals in these expressions are shown to be small. If we take for Z and S arbitrary functions with period $2\pi R$, satisfying equation 281, then equations 280 and 278 follow naturally. Thus equation 271 can be solved by successive approximations, starting with $Z_0=0$, $S_0=0$. At the first stage, take

$$L_1(Z_1,S_1)=\Lambda_1(Z_0,S_0),\qquad L_2(Z_1,S_1)=\Lambda_2(Z_0,S_0)\qquad \text{with } \int_0^{2\pi R}S_1(u)du=0;$$

at the second stage, take

$$L_1(Z_2,S_2)=\Lambda_1(Z_1,S_1),\qquad L_2(Z_2,S_2)=\Lambda_2(Z_1,S_1)\qquad \text{with } \int_0^{2\pi R}S_2(u)du=0,$$

. . . .

Finally we obtain the solution in unconditionally and uniformly convergent form

$$Z=Z_1+\sum_{k=2}^{\infty}(Z_k-Z_{k-1}),\qquad S=S_1+\sum_{k=2}^{\infty}(S_k-S_{k-1}).$$

Next, consider the free oscillation of the ring. Take $\mathfrak{M}=0$, and look for solutions of the form

$$\zeta=\zeta(s,t)=Z(s-R\delta t)=Z(u),\qquad \sigma=\sigma(s,t)=Z(s-R\delta t)=S(u) \tag{282}$$

where $Z(u)$, $S(u)$ are periodic with period $2\pi R$ in u. The equations

$$R^2\delta^2\frac{d^2Z}{du^2}+2R\delta\omega\frac{dS}{du}-\nu Z=0,$$

$$R^2\delta^2\frac{d^2S}{du^2}-2R\delta\omega\frac{dZ}{du}=0,$$

admit the three solutions (equation 274) with period $2\pi R$ only when δ^2 takes the value $\delta_0^2=4\omega^2-\nu=2\omega^2+\kappa\pi\mu/R=\kappa(2M+3\pi\mu R)/R^2$; that is 0, R; $R\cos u/R$, $(2R\omega/\delta_0)\sin(u/R)$; and $R\sin u/R$, $-(2R\omega/\delta_0)\cos(u/R)$.

Examine the solution for $\delta=\delta_0+\eta$ with a sufficiently small η. Our equations become

$$L^{(1)}(Z,S)\equiv R^2\delta_0^2\frac{d^2Z}{du^2}+2R\delta_0\omega\frac{dS}{du}-\nu Z$$
$$=-\eta\left(2\delta_0R^2\frac{d^2Z}{du^2}+2R\omega\frac{dS}{du}\right)-R^2\eta^2\frac{d^2Z}{du^2}+R\delta^2\left(\frac{dS}{du}\right)^2-2\delta\omega Z\frac{dS}{du}$$
$$+\delta^2Z\left(\frac{dS}{du}\right)^2-\frac{M\kappa}{R}\left(\frac{Z^2}{R^2}-\frac{Z^3}{R^3}-\ \dots\right)$$
$$+\kappa\int_C\mu'du'\left\{\frac{\partial}{\partial\nu}\log\frac{R}{\rho}-\frac{\partial}{\partial\nu_0}\log\frac{R}{\rho_0}\right\}\equiv\Lambda^{(1)}(Z,S),$$

$$L^2(Z^\circ S)\equiv R^2\delta_0^2\frac{d^2S}{du^2}-2R\delta_0\omega\frac{dZ}{du}$$
$$=-\eta\left(2\delta_0R^2\frac{d^2S}{du^2}-2R\omega\frac{dZ}{du}\right)-R^2\eta^2\frac{d^2S}{du^2}-R\delta^2Z\frac{d^2S}{du^2}$$
$$-2R\delta^2\frac{dS}{du}\frac{dZ}{du}+\kappa\int_C\mu'du'\left\{\frac{\partial}{\partial\tau}\log\frac{R}{\rho}-\frac{\partial}{\partial\tau_0}\log\frac{R}{\rho_0}\right\}\equiv\Lambda^{(2)}(Z,S).$$

We seek the solution of these equations such that

$$\int_0^{2\pi R}\left[\sin\frac{u}{R}\cdot Z(u)-\frac{2\omega}{\delta_0}\cos\frac{u}{R}\cdot S(u)\right]du=0,\qquad\int_0^{2\pi R}S(u)du=0.$$

Denote the Green function for

$$\frac{d^2Z}{du^2}-\frac{\nu}{\delta_0^2R^2}Z=0$$

by $\mathfrak{G}_0(\xi,\ u)$, such that $\mathfrak{G}_0(\xi,\ 2\pi R)=\mathfrak{G}_0(\xi,\ 0)$, $(\partial/\partial u)\mathfrak{G}_0(\xi,\ 2\pi R)=(\partial/\partial u)\mathfrak{G}_0(\xi,\ 0)$, $\mathfrak{G}_0(-\xi,-u)=\mathfrak{G}_0(\xi,\ u)$. Similarly to the former case, we obtain an integral equation

$$\mathrm{z}(\xi)=\int_E K_0(\xi,\ u)\mathrm{z}(u)du+\Psi_0(\xi).$$

The homogeneous equation has three null-solutions

$$w_1=0,\bar{A}R;\qquad w_2=\bar{B}R\cos\frac{u}{R},\ \bar{B}R\frac{2\omega}{\delta_0}\sin\frac{u}{R};\qquad w_3=\bar{B}R\sin\frac{u}{R},-\bar{B}R\frac{2\omega}{\delta_0}\cos\frac{u}{R};$$

where

$$\bar{A}=\frac{1}{2\pi R^3},\qquad\bar{B}=\frac{1}{\pi R^3}\frac{\delta_0^2}{4\omega^2+\delta_0^2}.$$

Form an integral equation with the kernel

$$K_0(\xi,\ u)\ -w_1(\xi)w_1(u)-w_2(\xi)w_2(u)-w_3(\xi)w_3(u)=N(\xi,\ u).$$

Put

$$\int_0^{2\pi R} \left(\cos\frac{u}{R} \cdot Z(u) + \frac{2\omega}{\delta_0} \sin\frac{u}{R} \cdot S(u) \right) du = d, \tag{283}$$

then our equations take the form

$$Z(\xi) - \bar{B}^{-2}R^2 \int_0^{2\pi R} \cos\frac{\xi - u}{R} Z(u)\, du$$

$$+ \int_0^{2\pi R} \left(\frac{2\omega}{\delta_0 R} \frac{\partial}{\partial u} \mathfrak{G}_0(\xi, u) - \bar{B}^2R^2 \frac{2\omega}{\delta_0} \sin\frac{\xi - u}{R} \right) S(u) du$$

$$= d\bar{B}^2R^2 \cos\frac{\xi}{R} - \frac{1}{\delta_0^2 R^2} \int_0^{2\pi R} \mathfrak{G}_0(\xi, u) \Lambda^{(1)} [Z(u), S(u)] du,$$

$$S(\xi) - \int_0^{2\pi R} \left[\frac{2\omega}{\delta_0 R} \frac{\partial}{\partial u} \mathfrak{G}_0(\xi, u) - \frac{2\omega}{\delta_0} \bar{B}^2R^2 \sin\frac{\xi - u}{R} \right] Z(u) du$$

$$- \int_0^{2\pi R} \left[\frac{\nu}{\delta_0^2 R^2} \mathfrak{G}_0(\xi, u) - \bar{A}^{-2}R^2 - \frac{4\omega^2}{\delta_0^2} \bar{B}^2R^2 \cos\frac{\xi - u}{R} \right] S(u) du$$

$$= \frac{2\omega}{\delta_0} \bar{B}^2R^2 d \sin\frac{\xi}{R} - \frac{1}{\delta_0^2 R^2} \int_0^{2\pi R} \mathfrak{G}_0(\xi, u) \Lambda^{(2)}(Z(u), S(u)) du,$$

with $Z(-\xi) = Z(\xi)$, $S(-\xi) = -S(\xi)$. Denote the resolvent of

$$\mathrm{Z}(\xi) = \int_E N(\xi, u) \mathrm{Z}(u) du + \Theta(\xi)$$

by $\mathfrak{H}(\xi, u)$; then the solution is

$$\mathrm{Z}(\xi) = \Theta(\xi) - \int_E \mathfrak{H}(\xi, u) \Theta(u) du.$$

Here we have taken the domain to be E, the double circuits of 0 to $2\pi R$. Develop the solution $Z(\xi)$, $S(\xi)$ in powers of d, η, and μ. Each term of the expansion contains d. Substitute these expansions for $Z(\xi)$, $S(\xi)$ in equation 283; then we can express η as a power series of d and μ. This is the bifurcation equation. These solutions are all periodic, and represent progressive waves. The configuration of the particles rotates with angular velocity $\omega + \delta_0$ with reference to the coordinate system fixed in space.

For other kinds of periodic solutions we consider $Z(u), S(u)$ to be of the form of equation 282 but with period $2\pi R/m$ $(m > 1)$. In order for them to have a period $2\pi R/m$, the value of δ^2 should be

$$\delta_1^2 = \frac{1}{m^2} \left(2\omega^2 + \frac{\kappa\pi\mu}{R} \right) = \frac{1}{m^2} \delta_0^2 = \frac{\kappa}{m^2R^2} (2M + 3\mu\pi R).$$

We obtain the solution by putting $\delta = \delta_1 + \eta^*$, just as in the previous case.

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APPENDIX A

Poincaré's Tidal Theory

EQUATIONS OF MOTION

Dynamical theory of the tidal oscillation of oceanic water has been treated by Hough (1897) and Goldsbrough (1928, 1929, 1930, 1931, 1933) by the method of forced oscillations. Poincaré (1896, 1903, 1910a, b) discussed the theory by use of Fredholm's theory of integral equations in the boundary value problem. The solution can be expanded in generalized Fourier series in the eigenfunctions of the boundary value problem, so that whatever the shape of the coast and the bottom of the ocean, the expansion will be carried out by numerical evaluation, possibly with electronic computers, and all possible modes of oscillation will be derived. Poincaré's theory was applied by Blondel (1912) and Chandon (1930) to the Red Sea, but not with much success. Jager (1916) considered an ocean bounded by a vertical coast and discussed the Green function of the problem and proposed Ritz's method of variation in accordance with Poincaré (1910b). Bertrand (1923) studied in detail the singularity of Poincaré's integral equation for a dynamical tide. Proudman (1913, 1914, 1916, 1924, 1928, 1932, 1933; Proudman and Mercer 1926, 1927; Proudman and Doodson 1924), with practical applications (1925–1929), based his theory on the quadratic form of an infinite number of variables in a manner similar to Hilbert's theory of Fredholm's linear integral equations.

Laplace derived the equations for tidal oscillation which can be reduced to a partial differential equation of the second order of the elliptic type. The coefficients become infinity at the critical latitude, as well as the integral appearing in Poincaré's integral equation. The difficulty can be avoided by taking Cauchy's principal value for the integral; also by iterating the kernel of Poincaré's integral equation, using Fredholm's procedure.

When the period of tide tends to infinity, it is called the *statical tide of the second kind.* The *tide of the first kind* does not depend on the depth of the ocean, as studied by Laplace. The long-period tide of the second kind and the dynamical diurnal and semidiurnal tides are studied in the present article. Proudman classified oceans in three classes by the eigenvalues of the proper oscillation (see the criticism by Fichot, 1938).

Suppose that the oceanic water is a perfect and incompressible fluid with unit density, and consider the forced oscillation under the attraction of the sun and moon during the uniform rotation ω of the earth. Let the coordinates of the earth's center in the space-fixed reference system be x, y, z and those of a point in the ocean referred to the earth's center be ξ, η, ζ. Denote by $\Pi(x, y, z, t)$ the potential due to the tidally deformed earth and by P the potential due to the sun and moon. Let p be the pressure in the oceanic water at (x, y, z); then the force is represented by grad $(\Pi + P - p)$. If the potential at the earth's center due to the sun and moon is P_0, then the force acting on an ocean molecule at (x, y, z) is grad Q, $Q = \Pi + (P - P_0) - p$. Denote by ω the rotational velocity of the earth and by (x_1, y_1, z_1) the coordinates referred to the earth's center in the reference frame rotating with the earth. Let δ be the distance from the rotation axis of the earth to the molecule in question, and u, v, w be the displacement of the molecule from its equilibrium position; then the equations of motion of an oceanic molecule are

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - 2\omega \frac{\partial u}{\partial t} &= \frac{\partial R}{\partial x_1}, \\ \frac{\partial^2 v}{\partial t^2} + 2\omega \frac{\partial v}{\partial t} &= \frac{\partial R}{\partial y_1}, \qquad R = \Pi + \frac{\omega^2 \delta^2}{2} + (P - P_0) - p, \\ \frac{\partial^2 w}{\partial t^2} &= \frac{\partial R}{\partial z_1}. \end{aligned}$$

Let the colatitude θ, the longitude ψ, and the radius vector ρ be the polar coordinates of the molecule referred to the rotating earth with the earth's center as origin, and U, V, W be the displacement of the molecule from its equilibrium respectively along the meridian, the latitude parallel, and the radius vector; then,

$$\begin{aligned} U &= u \cos\theta \cos\psi + v \cos\theta \sin\psi - w \sin\theta, \\ V &= -u \sin\psi + v \cos\psi, \\ W &= u \sin\theta \cos\psi + v \sin\theta \sin\psi + w \cos\theta, \end{aligned}$$

and the equations of motion take the form

$$\left.\begin{aligned} \frac{\partial^2 U}{\partial t^2} - 2\omega \cos\theta \frac{\partial V}{\partial t} &= \frac{\partial R}{\partial \theta}, \\ \frac{\partial^2 V}{\partial t^2} + 2\omega \cos\theta \frac{\partial U}{\partial t} + 2\omega \sin\theta \frac{\partial W}{\partial t} &= \frac{\partial R}{\sin\theta \partial\psi}, \\ \frac{\partial^2 W}{\partial t^2} - 2\omega \sin\theta \frac{\partial V}{\partial t} &= \frac{\partial R}{\partial \rho}, \end{aligned}\right\} \tag{A1}$$

Assume that the depth of the ocean $h(\theta, \psi)$ is small compared with the amplitude of the tidal oscillation. If there is no discontinuity in the oceanic depth as a function of θ, ψ, then $\partial h/\partial x$, $\partial h/\partial y$, or $\partial h/\partial\theta$, $\partial h/\partial\psi$ are small, and we see that w, $\partial w/\partial z$, $\partial w/\partial t$ are negligible. The potential Π is divided into the timely constant part for equilibrium Π', due to the earth's mass, and the variable part Π'', due to the part that is distorted tidally from equilibrium and is regarded as due to the simple layer of fluid disturbed over the equilibrium free surface Σ of the ocean; that is,

$$\Pi''(\theta, \psi) = -\int\int_{\Sigma} \frac{\zeta(\theta', \psi')d\sigma'}{r(\theta, \psi; \theta', \psi')}, \tag{A2}$$

where ζ is the negative displacement in the vertical direction evaluated at the point (θ', ψ') with surface element $d\sigma'$ and the distance r between the points (θ, ψ) and (θ', ψ'). Put

$$\Pi' + \frac{\omega^2\delta^2}{2} = G,$$

and let G_0 be the value of G at the equilibrium surface where g is the gravity acceleration; then,

$$G = G_0 - g\zeta.$$

Dropping the constant part, we obtain, at the free surface,

$$R = g\zeta + \Pi'' + (P - P_0). \tag{A3}$$

This gives the boundary condition at the free surface.

Since the displacement of water molecules should be tangential to the boundary of the ocean, a boundary condition is

$$Vd\theta - U \sin\theta d\psi = 0.$$

The equation for the boundary is $h(\theta, \psi) = 0$. If the ocean is limited by coasts with smooth variation of depth, then we take the general boundary condition at the boundary of the ocean to be

$$hVd\theta - hU \sin\theta d\psi = 0. \tag{A4}$$

The increase of the oceanic water through a curve C with arc element $(d\theta, \sin\theta d\psi)$ is

$$\int_C (hVd\theta - hU \sin\theta d\psi),$$

which is equal to

$$-\int\int_C \zeta d\sigma,$$

where

$$d\sigma = \sin\theta d\theta d\psi,$$

integrated over the area S enclosed by C. Hence

$$\int_C (hVd\theta - hU \sin\theta d\psi) = -\int\int_S \zeta \sin\theta d\theta d\psi,$$

or, by Green's formula,

$$\int\int_S \left[-\frac{\partial(hV)}{\partial\psi} - \frac{\partial(hU \sin\theta)}{\partial\theta}\right] d\theta d\psi = -\int\int_S \zeta \sin\theta d\theta d\psi.$$

Since the relation is valid for any closed curve C, we should have

$$\zeta \sin\theta = \frac{\partial (hU \sin\theta)}{\partial\theta} + \frac{\partial (hV)}{\partial\psi}, \tag{A5}$$

which is the equation of continuity. Equations A1 through A5 were derived by Laplace.

Denote by φ the inner angle at the moon M of the spherical triangle formed by M, the zenith Z, and the earth's north pole P on the celestial sphere; denote by χ the polar distance PM, by ρ the distance between the earth and the moon in space, by μ the mass of the moon, and by L the right ascension of the moon. Then the principal part of the disturbing action of the moon is

$$P - P_0 = \frac{\mu}{\rho^3}\frac{3\cos^2\varphi - 1}{2} + \ldots,$$

or

$$P - P_0 = \frac{3\mu}{4\rho^3}\sin^2\theta \sin^2\chi \cos 2(\omega t + \psi - L)$$

$$+ \frac{3\mu}{4\rho^3}\sin 2\theta \sin 2\chi \cos(\omega t + \psi - L)$$

$$+ \frac{\mu}{\rho^3}\frac{3\cos^2\theta - 1}{2}\frac{3\cos^2\chi - 1}{2},$$

$$P - P_0 = \sin^2\theta \cdot e^{2i\psi}\sum A_2 e^{i(2\omega+\nu)t} + \sin^2\theta \cdot e^{-2i\psi}\sum A_2' e^{-i(2\omega+\nu)t}$$

$$+ \sin 2\theta \cdot e^{i\psi}\sum A_1 e^{i(\omega+\nu)t} + \sin 2\theta \cdot e^{-i\psi}\sum A_1' e^{-i(\omega+\nu)t}$$

$$+ \frac{3\cos^2\theta - 1}{2}\sum A_0 e^{i\nu t} + \frac{3\cos^2\theta - 1}{2}\sum A_0' e^{-i\nu t}$$

where A and ν are given by the perturbation theory when $\nu \ll \omega$. The variations of ρ, χ, L are periodic, with longer periods than the period of ω.

The first line contains the terms with periods nearly equal to 12 hours, which are called the *semidiurnal tide;* the second line contains the terms with periods nearly equal to 24 hours, which are called the *diurnal tide.* The third line contains terms of long periods. These three kinds of tide constitute the *dynamical tide.* The tide due to the terms of infinitely long period is called the *statical tide.* Since the equations of tide are linear, we can add the effect of each term by solving separately the differential equations with each one of the terms of perturbation. Since the perturbation is decomposed into complex terms, the forced oscillation due to one such term is called an *isochronous complex forced oscillation.* We consider a term $F(\theta, \psi)e^{i\alpha t}$ as the representative.

We obtain such an oscillation by putting

$$U, V, R, \zeta \propto e^{i\alpha t}.$$

Equations A1 are transformed by writing $R = \alpha^2\Phi$ into

$$-\alpha U - 2i\omega V \cos\theta = \alpha \frac{\partial \Phi}{\partial \theta},$$

$$-\alpha V + 2i\omega U \cos\theta = \alpha \frac{\partial \Phi}{\sin\theta \partial \psi},$$

$$\zeta \sin\theta = \frac{\partial (hU \sin\theta)}{\partial \theta} + \frac{\partial (hV)}{\partial \psi},$$

$$\alpha^2 \Phi = g\zeta + \Pi'' + F(\theta, \psi). \tag{A6}$$

Solving these, we obtain

$$U = \frac{\alpha^2}{4\omega^2 \cos^2\theta - \alpha^2} \frac{\partial \Phi}{\partial \theta} - \frac{2i\alpha\omega \cos\theta}{4\omega^2 \cos^2\theta - \alpha^2} \frac{\partial \Phi}{\sin\theta \partial \psi},$$

$$V = \frac{\alpha^2}{4\omega^2 \cos^2\theta - \alpha^2} \frac{\partial \Phi}{\sin\theta \partial \psi} + \frac{2i\alpha\omega \cos\theta}{4\omega^2 \cos^2\theta - \alpha^2} \frac{\partial \Phi}{\partial \theta}.$$

The continuity equation is now

$$\zeta \sin\theta = \frac{\partial}{\partial \theta}\left(h_1 \sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(h_1 \frac{\partial \Phi}{\sin\theta \partial \psi}\right) + \frac{\partial \Phi}{\partial \theta}\frac{\partial h_2}{\partial \psi} - \frac{\partial \Phi}{\partial \psi}\frac{\partial h_2}{\partial \theta},$$

$$h_1 = \frac{h\alpha^2}{4\omega^2 \cos^2\theta - \alpha^2}, \qquad h_2 = \frac{2i\omega\alpha h \cos\theta}{4\omega^2 \cos^2\theta - \alpha^2},$$

with the boundary condition

$$\alpha \frac{\partial \Phi}{\partial n} + 2i\omega \cos\theta \frac{\partial \Phi}{\partial s} = 0,$$

where n, s are the normal and the tangent to the boundary curve, respectively.

These equations become critical at

$$4\omega^2 \cos^2\theta - \alpha^2 = 0; \qquad \text{i.e., } \cos\theta = \pm \frac{\alpha}{2\omega},$$

which is called the *critical latitude*. For a semidiurnal tide, $\alpha \approx 2\omega$, and the critical latitude is near the pole; for a diurnal tide, $\alpha \approx \omega$, and the critical latitude is near ± 30 degrees. The critical latitude occurs because we neglect centrifugal force. If we transfer the centrifugal-force term $\omega^2\delta^2/2$ in R to the left-hand side and treat it in a similar manner, a new set of equations with different critical latitude such that $\cos^2\theta = [\alpha^2(\alpha^2+\omega^2)]/[\omega^2(3\alpha^2-\omega^2)]$ will appear, where the denominator is always positive for the semidiurnal and the diurnal tides.

STATICAL TIDE AND FREDHOLM'S EQUATION

Consider the long-period tide of the first kind due to the terms of planetary perturbations

$$\frac{3\cos^2\theta - 1}{2} \sum \frac{\mu}{\rho^3} \frac{3\cos^2\chi - 1}{2}$$

where the sum can be written

$$C=\sum \frac{\mu}{\rho^3}\frac{3\cos^2\chi-1}{2}=\sum (A\cos\alpha t+B\sin\alpha t).$$

Hough (1897) classified the statical tide in two kinds—the first kind deduced from the equilibrium theory and the second kind obtained by putting $\alpha=0$ in the equations for dynamical tide. We consider the long-period tide of the first kind. We neglect in equation A1 the accelerations $\partial^2U/\partial t^2$, $\partial^2V/\partial t^2$ and the centrifugal force $2\omega\cos\theta\cdot\partial V/\partial t$, $-2\omega\cos\theta\cdot\partial U/\partial t$; then we have $\partial R/\partial\theta=\partial R/\partial\psi=0$ from equation A1 and obtain $R=k(t)$ independent of θ, ψ. Hence the equation for statical tide is

$$g\zeta(\theta,\psi)-\iint_\Sigma \frac{\zeta(\theta',\psi')d\sigma'}{r(\theta,\psi;\theta',\psi')}+C\frac{3\cos^2\theta-1}{2}=k$$

on the ocean with surface Σ and

$$\zeta=0$$

on the continent, where

$$\iint_\Sigma \zeta d\sigma=0$$

Writing

$$\frac{1}{g}=\lambda,\qquad \frac{k}{g}-\frac{C}{g}\frac{3\cos^2\theta-1}{2}=\chi(\theta)$$

and $\epsilon(\theta,\psi)=1$ on the ocean and $\epsilon(\theta,\psi)=0$ on the continent, we obtain

$$\zeta(\theta,\psi)=\lambda\iint \frac{\epsilon(\theta,\psi)\epsilon(\theta',\psi')\zeta(\theta',\psi')d\sigma'}{r(\theta,\psi;\theta',\psi')}+\epsilon(\theta,\psi)\chi(\theta). \tag{A7}$$

The kernal of equation A7 becomes infinite of the first order if the two points (θ,ψ) and (θ',ψ') coincide. But, since the integral is double, the iterated kernel is finite (Lalesco, 1910), and Fredholm's method of solution can be applied. Denote the resolving kernel by $K(\theta,\psi;\theta',\psi';\lambda)$; then the solution is

$$\zeta(\theta,\psi)=\epsilon(\theta,\psi)\chi(\theta)+\lambda\iint K(\theta,\psi;\theta',\psi';\lambda)\epsilon(\theta',\psi')\chi(\theta')d\sigma', \tag{A8}$$

Since the kernel is symmetrical, there exists at least one real and simple pole of the resolvent.

The potential Π due to the part of the oceanic water which is displaced from the equilibrium state can be regarded as the potential due to the surface distribution of ζ:

$$\zeta(\theta,\psi)=\lambda\iint_\Sigma \frac{\zeta(\theta',\psi')d\sigma'}{r(\theta,\psi;\theta',\psi')}+\chi(\theta); \tag{A9}$$

Function Π is a meromorphic function with the same pole as the pole of ζ and K. This potential Π satisfies at the surface (Poincaré, 1899)

$$2\frac{\partial \Pi}{\partial \rho}+\Pi=-4\pi\zeta, \tag{A10}$$

where ρ is the radius vector and $\zeta=-\lambda\Pi+\chi$ on the ocean and $\zeta=0$ on the continent, so that $\zeta=\epsilon(-\lambda\Pi+\chi)$ on the whole surface; Π is a harmonic function defined by $\Delta\Pi=0$ inside the earth and

$$2\frac{\partial \Pi}{\partial \rho}+\Pi=4\pi\epsilon(\lambda\Pi-\chi)$$

on the surface.

Let ζ_0 be one of the poles of $\zeta(\lambda)$ and consider

$$\zeta_0(\theta,\psi)=\lambda_0\iint\frac{\epsilon(\theta,\psi)\epsilon(\theta',\psi')\zeta_0(\theta',\psi')\,d\sigma'}{r(\theta,\psi;\theta',\psi')},$$

The potential Π_0 due to the displaced mass ζ_0 satisfies on the surface

$$2\frac{\partial \Pi_0}{\partial \rho}+\Pi_0=4\pi\epsilon\lambda_0\Pi_0.$$

From this we obtain

$$2\iint\Pi_0\frac{\partial \Pi_0}{\partial \rho}\,d\sigma=\iint\Pi_0^2(4\pi\epsilon\lambda_0-1)\,d\sigma,$$

where, by Green's theorem,

$$\iint\Pi_0\frac{\partial \Pi_0}{\partial \rho}\,d\sigma=\iiint\left[\left(\frac{\partial \Pi_0}{\partial x}\right)^2+\left(\frac{\partial \Pi_0}{\partial y}\right)^2+\left(\frac{\partial \Pi_0}{\partial z}\right)^2\right]dxdydz\geqq 0.$$

Hence,

$$\iint\Pi_0^2(4\pi\epsilon\lambda_0-1)\,d\sigma\geqq 0.$$

Thus λ_0 should be zero or positive, and we obtain

$$4\pi\lambda_0\geqq 1+\left[\iint_{continent}\Pi_0^2 d\sigma\Big/\iint_{ocean}\Pi_0^2 d\sigma\right]\geqq 1.$$

Consequently the first pole, if it exists, is positive such that $\lambda_0^2\geqq 1/4\pi$.

Since $\Pi(\lambda)$ is a meromorphic function of λ, it can be expanded

$$\Pi=\Pi_0+\lambda\Pi_1+\ldots+\lambda^n\Pi_n+\ldots,$$

and satisfies

$$2\frac{\partial \Pi}{\partial \rho}+\Pi=4\pi\epsilon\lambda\Pi-4\pi\epsilon\chi.$$

Substituting the expansion of Π in the latter equation and equating the coefficients of various powers of λ, we obtain

$$2\frac{\partial \Pi_0}{\partial \rho}+\Pi_0=-4\pi\epsilon\chi,$$
$$2\frac{\partial \Pi_n}{\partial \rho}+\Pi_n=4\pi\epsilon\Pi_n \qquad (n=1, 2, \ldots). \tag{A11}$$

Denote Schwarz's constants by

$$W_{p,q}=\int\int_\epsilon \Pi_p\Pi_q d\sigma. \tag{A12}$$

Green's formula gives, on the other hand,

$$2\int\int\left(\Pi_p\frac{\partial \Pi_q}{\partial \rho}-\Pi_q\frac{\partial \Pi_p}{\partial \rho}\right)d\sigma=0,$$

or, by equation A11,

$$\int\int_\epsilon (\Pi_p\Pi_{q-1}-\Pi_q\Pi_{r-1})d\sigma=0.$$

Hence,

$$W_{p,q-1}=W_{p-1,q},$$

or

$$W_{p,q}=W_{r-1,q+1}=W_{r-2,q+2}=\ldots=W_{0,p+q}.$$

This gives a recurrent formula for $W_{p,q}$. From equation A11, it can be shown that

$$W_{2p}\equiv W_{0,2p}=W_{p,p}=\int\int_\epsilon \Pi_p^2 d\sigma>0,$$

$$W_{2p-1}\equiv W_{0,2p-1}=W_{p,p-1}=\int\int_\epsilon \Pi_r\Pi_{r-1}d\sigma=\frac{1}{2\pi}\int\int\Pi_p\frac{\partial \Pi_p}{\partial \rho}d\sigma+\frac{1}{4\pi}\int\int\Pi_p^2 d\sigma,$$

which shows that $W_{0,n}>0$.

Further, by forming

$$\int\int_\epsilon (\alpha\Pi_n+\beta\Pi_{n+1})^2 d\sigma>0,$$

we see that the quadratic forms

$$\alpha^2W_{2n-1}+2\alpha\beta W_{2n}+\beta^2W_{2n+1}, \qquad \alpha^2W_{2n}+2\alpha\beta W_{2n+1}+\beta^2W_{2n+2},$$

are both positive definite, and we have

$$\frac{W_{2n}}{W_{2n-1}}<\frac{W_{2n+1}}{W_{2n}}<\frac{W_{2n+2}}{W_{2n+1}}<\ldots.$$

Integrating the expansion of Π term-by-term, we obtain

$$W(\lambda)=W_0+\lambda W_1+\ldots+\lambda^n W_n+\ldots,$$

and

$$\frac{W_{2n}}{W_{2n-1}} \leqq 4\pi \left[\iint \epsilon \Pi_n^2 \, d\sigma \Big/ \iint \Pi_n{}^2 d\sigma \right] \leqq 4\pi.$$

Hence

$$0 < \lim_{n\to\infty} \frac{W_n}{W_{n-1}} = \frac{1}{\lambda_1} \leqq 4\pi.$$

The radius of convergence of $W(\lambda)$ is accordingly equal to λ_1.

It can be shown that λ_1 is the first pole of $\Pi(\lambda)$ and that the radius of convergence of Π in powers of λ is λ_1. In fact, from equation A11 with $n=n$, we have

$$\Pi_n(\rho, \theta, \psi) = \iint \frac{\epsilon(\theta', \psi')\Pi_{n-1}(\theta', \psi')d\sigma'}{r(\rho, \theta, \psi; \theta', \psi')}.$$

Since the factor of Π_{n-1} in the integrand is singular, we iterate

$$\Pi_n(\rho, \theta, \psi) = \iint \frac{\epsilon(\theta', \psi')d\sigma'}{r(\theta, \psi; \theta', \psi')} \iint \frac{\epsilon(\theta'', \psi'')\Pi_{n-2}(\theta'', \psi'')}{r(\theta', \psi'; \theta'', \psi'')} d\sigma''$$

$$= \iint \epsilon(\theta'', \psi'')\Pi_{n-1}(\theta'', \psi'')d\sigma'' \iint \frac{\epsilon(\theta', \psi')d\sigma'}{r(\theta', \psi'; \theta'', \psi'')r(\rho, \theta, \psi; \theta', \psi')}.$$

If the three points (θ, ψ), (θ', ψ'), (θ'', ψ'') coincide, then the second integral becomes infinite, at most, logarithmically (Heywood and Fréchet, 1912). Using Schwarz's inequality, we have, from the foregoing equation,

$$[\Pi_n(\rho, \theta, \psi)]^2 \leqq \iint [\epsilon(\theta'', \psi'')\Pi_{n-2}(\theta'', \psi'')]^2 d\sigma'' \times \iint I^2 d\sigma',$$

where

$$I = \frac{\epsilon(\theta', \psi')}{r(\theta', \psi'; \theta'', \psi'')r(\rho, \theta, \psi; \theta', \psi')},$$

or, from $\epsilon^2 = \epsilon$ and

$$\iint [\epsilon(\theta'', \psi'')\Pi_{n-2}(\theta'', \psi'')]^2 d\sigma'' = \iint \epsilon(\theta'', \psi'')[\Pi_{n-2}(\theta'', \psi'')]^2 d\sigma'' = W_{2n-4},$$

we obtain

$$[\Pi_n(\rho, \theta, \psi)]^2 \leqq W_{2n-4}K^2, \qquad K^2 = \iint I^2 d\sigma'.$$

Hence

$$|\chi^n \Pi_n(\rho, \theta, \psi)| < K|\lambda^n| \sqrt{W_{2n-4}},$$

so that the series for $\Pi(\rho, \theta, \psi; \lambda)$ converges uniformly if the series for $\lambda^n \sqrt{W_{2n-4}}$ does. The latter series is convergent in a circle of radius λ_1. Consequently, the radius of convergence and the first pole of $\Pi(\lambda)$ is λ_1.

Now we expand Π in the form

$$\Pi=\frac{\Pi^{(1)}}{1-\frac{\lambda}{\lambda_1}}+v_0+\lambda v_1+\ .\ .\ .\ +\lambda^n v_n+\ .\ .\ ..$$

By a similar procedure, we can see that the second pole $\lambda_2>\lambda_1$ and that there are an infinite number of discrete poles λ_1, λ_2, λ_3, . . . which are all positive and at least equal to $1/4\pi$. Note that this conclusion does not depend on the form of the continent. Thus the problem of statical tide of the first kind has been solved.

FREDHOLM'S EQUATION FOR DYNAMICAL TIDE

For convenience, transform spherical coordinates ρ, θ, ψ, into rectangular coordinates x, y on the geographical map:

$$ds^2=d\theta^2+\sin^2\theta d\psi^2=\frac{1}{k^2}\,(dx^2+dy^2),$$

where k stands for the similitude ratio. Dividing u and v by k and denoting the result simply by u and v, we obtain the equations

$$\frac{\partial^2 u}{\partial t^2}-2\omega\cos\theta\frac{\partial v}{\partial t}=\frac{\partial R}{\partial x},\qquad \frac{\partial^2 v}{\partial t^2}+2\omega\cos\theta\frac{\partial u}{\partial t}=\frac{\partial R}{\partial y},$$

and the equation of continuity

$$\left.\begin{aligned}\frac{\zeta}{k}&=\frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y},\\ \Pi''&=-\iint\frac{\zeta' dx' dy'}{k'^2 r},\end{aligned}\right\}\tag{A6a}$$

the boundary condition being $hudy-hvdx=0$; or

$$\left.\begin{aligned}&\frac{\zeta}{k^2}=\frac{\partial}{\partial x}\left(h_1\frac{\partial\Phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(h_2\frac{\partial\Phi}{\partial y}\right)+\frac{\partial\Phi}{\partial x}\frac{\partial h_2}{\partial y}-\frac{\partial\Phi}{\partial y}\frac{\partial h_2}{\partial x},\\ &h_1=\frac{h\alpha^2}{4\omega^2\cos^2\theta-\alpha^2},\qquad h_2=\frac{2i\omega\alpha h\cos\theta}{4\omega^2\cos^2\theta-\alpha^2},\end{aligned}\right\}\tag{A6b}$$

with the boundary condition

$$\alpha\frac{\partial\Phi}{\partial n}+2i\omega\cos\theta\frac{\partial\Phi}{\partial s}=0.$$

Dividing equation (A6b) for ζ by h_2, we obtain

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial\Phi}{\partial x}\frac{1}{h_1}\left(\frac{\partial h_1}{\partial x}+\frac{\partial h_2}{\partial y}\right)+\frac{\partial\Phi}{\partial y}\frac{1}{h_1}\left(\frac{\partial h_1}{\partial y}-\frac{\partial h_2}{\partial x}\right)=\frac{\zeta}{k^2 h_1}.$$

The equation of the surface in equation (A6)

$$\alpha^2\Phi = g\zeta + \Pi'' + F(x, y)$$

is transformed to

$$\zeta(x, y) = \Psi(x, y) + \frac{\alpha^2}{g}\Phi(x, y) + \frac{\alpha^2}{g^2}\int\int_\Sigma K\left(x, y; x', y'; \frac{1}{g}\right)\Phi(x', y')\ \frac{dx'dy'}{k'^2},$$

where

$$\Psi(x, y) = -\frac{F(\theta, \psi)}{g} - \frac{1}{g^2}\int\int_\Sigma K\left(\theta, \psi; \theta', \psi'; \frac{1}{g}\right)F(\theta', \psi')d\sigma'$$

with the same kernel K. This is the integro-differential equation for dynamical tide. Equating $\zeta(x, y)$ of these equations, we arrive at

$$\Delta\Phi + a\frac{\partial\Phi}{\partial x} + b\frac{\partial\Phi}{\partial y} + c\Phi + e\int\int_\Sigma K'(x, y; x', y')\Phi(x', y')dx'dy' = f, \tag{A13}$$

where

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta, \qquad \frac{1}{h_1}\left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y}\right) = a, \qquad \frac{1}{h_1}\left(\frac{\partial h_1}{\partial y} - \frac{\partial h_2}{\partial x}\right) = b,$$

$$-\frac{\alpha^2}{gk^2h_1} = c, \qquad -\frac{\alpha^2}{g^2k^2h_1} = e, \qquad \frac{\Psi(x, y)}{k^2h_1} = f,$$

$$\frac{K\left(x, y; x', y'; \frac{1}{g}\right)}{k'^2} = K'(x, y; x', y').$$

On the contour C of the ocean, we have the boundary condition

$$\alpha\frac{\partial\Phi}{\partial n} + 2i\omega\cos\theta\frac{\partial\Phi}{\partial s} = 0. \tag{A14}$$

Consider the Green function $G(x, y; \xi, \eta)$ defined by the following conditions:
(1) the function $G_1(x, y; \xi, \eta)$ such that

$$G_1(x, y; \xi, \eta) = \log\frac{1}{r} - G(x, y; \xi, \eta), \qquad r^2 = (x-\xi)^2 + (y-\eta)^2,$$

for any point x, y inside the domain Σ is harmonic; i.e., $\Delta G = 0$ inside Σ.
(2) G satisfies

$$\alpha\frac{\partial G}{\partial n} + 2i\omega\cos\theta\frac{\partial G}{\partial s} = 0$$

along the contour C of Σ. It can be shown that the solution of equations A13 and A14 is given by the function Φ (x, y) defined by

$$\Phi(x, y)-\frac{1}{2\pi}\int_C (a'\alpha'+b'\beta')G\Phi(\xi,\eta)ds'$$
$$+\frac{1}{2\pi}\iint_\Sigma\left[\frac{\partial(a'G)}{\partial\xi}+\frac{\partial(b'G)}{\partial\eta}-c'G\right]\Phi(\xi, \eta)d\xi d\eta$$
$$-\frac{1}{2\pi}\iint_\Sigma e'Gd\xi d\zeta\iint_\Sigma K'(\xi, \eta; x', y')\Phi(x', y')dx'dy'$$
$$=-\frac{1}{2\pi}\iint_\Sigma f'Gd\xi d\eta, \tag{A15}$$

where a primed function is the function in which x, y are replaced by ξ, η, and where α', β' denote the direction cosines of the inward normal of the contour C. Thus the integro-differential equation A13 for dynamical tide is led to a Fredholm equation containing simple, double, and quadruple integrals.

Our first problem is to derive the Green function $G(x, y; \xi, \eta)$, that is, derive the function G so as to satisfy

$$\alpha\frac{\partial G}{\partial n}+2i\omega\cos\theta\frac{\partial G}{\partial s}=a\frac{\partial\log\frac{1}{r}}{\partial n}+2i\omega\cos\theta\frac{\partial\log\frac{1}{r}}{\partial s}$$

along C and to be harmonic inside Σ. In other words, we are to establish the existence of the function $V(x, y)$ which is harmonic inside Σ and satisfies the condition

$$\alpha\frac{\partial V}{\partial n}+2i\omega\cos\theta\frac{\partial V}{\partial s}=\chi(s) \tag{A16}$$

along the boundary C.

It is known that the logarithmic potential

$$V(x, y)=\int_C \rho(s')\log\frac{1}{r}\cdot ds', \qquad r=\sqrt{[x-\xi(s')]^2+[y-\eta(s')]^2},$$

where $\rho(s')$ is a continuous function of s', a curvilinear abscissa of an arbitrary point from the fixed point on C as origin, satisfies the following conditions:

(1) it is continuous in x, y for all points at finite distances in the plane,
(2) it is harmonic, so that $\Delta V=0$ at all points except the points on C,
(3) the inward normal $\partial V/\partial n_i$ and the outward normal $\partial V/\partial n_e$ satisfy

$$\frac{\partial V}{\partial n_i}=-\pi\rho(s)+\int_C\rho(s')\frac{\cos\psi}{r}ds', \qquad \frac{\partial V}{\partial n_e}=\pi\rho(s)+\int_C\rho(s')\frac{\cos\psi}{r}ds',$$

where ψ is the angle between the inward normal and the vector r, and the principal value of Cauchy's integral

$$\int_C{}'\rho(s')\frac{\sin\psi}{r}ds'\rightarrow\frac{\partial V}{\partial s}$$

converges to the tangential derivative uniformly if the radius of curvature of C is larger than a fixed number and $\rho(s)$ has its derivative. Such a function $\rho(s)$ satisfies

$$-\alpha\pi\rho(s)+\alpha\int_C \rho(s')\frac{\cos\psi}{r}\,ds'+2i\omega\cos\theta\int_C{}' \rho(s')\frac{\sin\psi}{r}\,ds'=\chi(s). \tag{A17}$$

This is an integral equation of Fredholm's type, but one of the integrals is replaced by Cauchy's principal value.

Let x, y be two variables in the complex plane, $f(y)$ be a function of y, and C be an arbitrary arc of a curve containing the points x and y. Exclude two arcs xa, xb of equal length on the curve on both sides of the point x. An ordinary integral

$$F(x,h)=\int_{C-ab}\frac{f(y)\,dy}{y-x},$$

which is a function of h, tends to a finite limit as $h\to 0$; then the limit is Cauchy's principal value denoted by

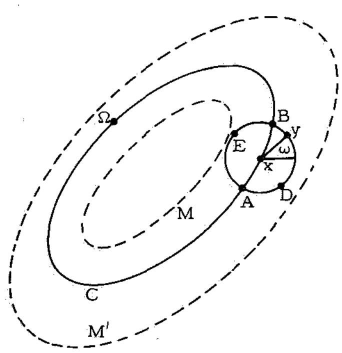
$$\int_C{}'\frac{f(y)\,dy}{y-x}.$$

Assume that $f(y)$ is holomorphic in a connected domain D, and construct two closed curves M and M' on both sides of the closed curve C with a unique tangent at each of its points. Describe a small circle $ADBE$ with radius h and center x. Let Ω be an arbitrary point on the closed curve C. We have

$$\frac{S}{2}=\frac{1}{2}\int_M\frac{f(y)}{y-x}\,dy+\frac{1}{2}\int_{M'}\frac{f(y)}{y-x}\,dy$$

$$=\frac{1}{2}\left(\int_{B\Omega A}+\int_{AEB}+\int_{B\Omega A}+\int_{ADB}\right)\frac{f(y)\,dy}{y-x}$$

$$=\int_{B\Omega A}\frac{f(y)\,dy}{y-x}+\frac{1}{2}\left(\int_{AEB}+\int_{ADB}\right)\frac{f(y)\,dy}{y-x},$$



Hence,

$$F(x,h)=\int_{B\Omega A}\frac{f(y)\,dy}{y-x}=\frac{S}{2}-\frac{1}{2}\left(\int_{AEB}+\int_{ADB}\right)\frac{f(y)\,dy}{y-x}.$$

Put

$$y=x+he^{i\omega},\qquad f(y)=f(x)+Ah;$$

then

$$\int_{AEB}\frac{f(y)\,dy}{y-x}=if(x)\int_{AEB}d\omega+ih\int_{AEB}A\,d\omega,$$

and

$$\lim_{h\to 0}\int_{AEB}\frac{f(y)\,dy}{y-x}=if(x)\lim_{h\to 0}(-B\hat{E}A).$$

Similarly,

$$\lim_{h\to 0}\int_{ADB}\frac{f(y)\,dy}{y-x}=if(x)\lim_{h\to o}(A\hat{D}B),$$

where $\lim\limits_{h\to 0}(A\hat{D}B-B\hat{E}A)=0$ if curve C has a unique tangent at each point. Thus the principal value defines a function $F(x)$ which is holomorphic at each point of C:

$$F(x)=\int_C'\frac{f(y)\,dy}{y-x}=\frac{1}{2}\int_M\frac{f(y)\,dy}{y-x}+\frac{1}{2}\int_{M'}\frac{f(y)\,dy}{y-x}.$$

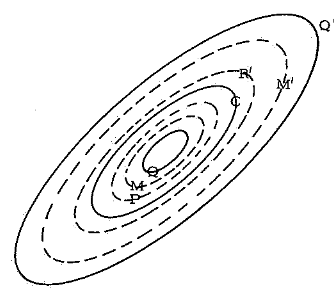
It can be proved that

$$\int_C'\frac{f(y)\,dy}{y-x}=\int_{M'}\frac{f(y)\,dy}{y-x}-i\pi f(x),\qquad \int_C'\frac{f(y)\,dy}{y-x}=\int_M\frac{f(y)\,dy}{y-x}+i\pi f(x).$$

Let $f_0(x)$ be holomorphic in the band domain D enclosed by two closed curves Q, Q' on each side of a closed simple curve C and outside of M, M'. Similarly, draw two closed curves P, P' respectively between C, M and C, M'. Let $A(x, y)$, $B(x, y)$ be functions of two complex variables x, y, which are holomorphic, while x, y vary in D. Put

$$f_1(x)=\int_C'\frac{A(x, y)}{y-x}f_0(y)\,dy,$$

$$f_2(x)=\int_C'\frac{B(x, y)}{y-x}f_1(y)\,dy.$$



Then,

$$f_1(y)=\frac{1}{2}\int_M\frac{A(y, z)}{z-y}f_0(z)\,dz+\frac{1}{2}\int_{M'}\frac{A(y, z)}{z-y}f_0(z)\,dz,$$

$$f_2(x)=\frac{1}{2}\int_P\frac{B(x, y)}{y-x}f_1(y)\,dy+\frac{1}{2}\int_{P'}\frac{B(x, y)}{y-x}f_1(y)\,dy,$$

and

$$f_2(x)=\frac{1}{4}\int_P\frac{B(x, y)}{y-x}\,dy\int_M\frac{A(y, z)}{z-y}f_0(z)\,dz+\frac{1}{4}\int_P\frac{B(x, y)}{y-x}\,dy\int_{M'}\frac{A(y, z)}{z-y}f_0(z)\,dz$$
$$+\frac{1}{4}\int_{P'}\frac{B(x, y)}{y-x}\,dy\int_M\frac{A(y, z)}{z-y}f_0(z)\,dz+\frac{1}{4}\int_{P'}\frac{B(x, y)}{y-x}\,dy\int_{M'}\frac{A(y, z)}{z-y}f_0(z)\,dz.$$

The four double integrals are expressed after computation by

$$I_1=\int_M f_0(z)\,dz\int_Q\frac{B(x, y)}{y-x}\frac{A(y, z)}{z-y}\,dy-2\pi i\int_M\frac{B(x, z)A(z, z)}{z-x}f_0(z)\,dz,$$

$$I_2=\int_{M'}f_0(z)\,dz\int_Q\frac{B(x, y)}{y-x}\frac{A(y, z)}{z-y}\,dy,$$

$$I_3=\int_M f_0(z)\,dz\int_{Q'}\frac{B(x, y)}{y-x}\frac{A(y, z)}{z-y}\,dy,$$

$$I_4=\int_{M'}f_0(z)\,dz\int_{Q'}\frac{B(x, y)}{y-x}\frac{A(y, z)}{z-y}\,dy+z\pi i\int_{M'}\frac{B(x, z)A(z, z)}{z-x}f_0(z)\,dz.$$

Thus

$$f_2(x)=\sum_1+\sum_2,$$

where

$$\sum_1=\frac{\pi}{2}i\int_{M'}\frac{B(x,z)A(z,z)}{z-x}f_0(z)dz-\frac{\pi}{2}i\int_M\frac{B(x,z)A(z,z)}{z-x}f_0(z)dz,$$

or, by computing the residue at $z=x$ as x varies on C,

$$\sum_1=-\pi^2A(x,x)B(x,x)f_0(x),$$

and

$$\sum_2=\frac{1}{4}\int_M f_0(z)dz\int_Q\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy+\frac{1}{4}\int_M f_0(z)dz\int_{Q'}\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy$$
$$+\frac{1}{4}\int_{M'}f_0(z)dz\int_Q\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy+\frac{1}{4}\int_{M'}f_0(z)dz\int_{Q'}\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy.$$

The integral

$$\int_C'\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy,$$

in which x, y, z vary on the same curve C, is equal to

$$BA(x,z)=\frac{1}{2}\int_{Q+Q'}\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy,$$

which is holomorphic in the ring domain between Q and Q'. If the singular integral exists, then

$$BA(x,z)=\int_C'\frac{B(x,y)}{y-x}\frac{A(y,z)}{z-y}dy,$$

and

$$\sum_2=\int_{M+M'}BA(x,z)f_0(z)dz.$$

Since $BA(x,z)$ and $f_0(z)$ are holomorphic in the ring domain between Q and Q', each of the contours M and M' can be deformed so that both coincide with C. Thus,

$$\sum_2=\int_C BA(x,z)f_0(z)dz.$$

We obtain a fundamental formula (Poincaré, 1910b),

$$\int_C'\frac{B(x,y)}{y-x}dy\int_C'\frac{A(y,z)}{z-y}f_0(z)dz=-\pi^2B(x,x)A(x,x)f_0(x)+\int_C BA(x,z)f_0(z)dz. \tag{A18}$$

Suppose that the function $A(x,y)$, $B(x,y)$, $f_0(x)$ of real variables admit period Ω in each of the variables and are holomorphic as x, y describe a small ring domain abutting on the closed curve C with unique tangent at each point. Put

$$u=e^{2\pi ix/\Omega}, \qquad v=e^{2\pi iy/\Omega}, \qquad w=e^{2\pi iz/\Omega},$$

and consider

$$f_1(x)=\int_0^{\prime\,\Omega} A(x,y)\,\frac{ie^{2\pi ix/\Omega}}{e^{2\pi iy/\Omega}-e^{2\pi ix/\Omega}}\,f_0(y)\,dy,$$

$$f_2(x)=\int_0^{\prime\,\Omega} B(x,y)\,\frac{ie^{2\pi ix/\Omega}}{e^{2\pi iy/\Omega}-e^{2\pi ix/\Omega}}\,f_1(y)\,dy.$$

It can be shown that

$$\begin{aligned} f_2(x) = &-\frac{\Omega^2}{4}B(x,x)A(x,x)f_0(x)\\ &+\int_0^{\Omega} f_0(z)\,dz\int_0^{\prime\,\Omega} B(x,y)A(y,z)\,\frac{ie^{2\pi ix/\Omega}}{e^{2\pi iy/\Omega}-e^{2\pi ix/\Omega}}\,\frac{ie^{2\pi iy/\Omega}}{e^{2\pi iz/\Omega}-e^{2\pi iy/\Omega}}\,dy. \end{aligned} \tag{A19}$$

Let $A(x, y)$ be an arbitrary point on C such that $x=f(t)$, $y=\varphi(t)$, and A describe a pole $y=x$ as the only singularity; i.e.,

$$M(x,y) = M_0(x,y) + \frac{M_1(x,y)}{y-x},$$

where $M_0(x, y)$, $M_1(x, y)$ are holomorphic. Then, by computation of the residue of $M_1(x)$, we obtain

$$\frac{2\pi}{\Omega}M_1(x) = \lim_{y\to x}\left[\frac{e^{2\pi iy/\Omega}-e^{2\pi ix/\Omega}}{ie^{2\pi ix/\Omega}}\,M(x,y)\right]. \tag{A20}$$

The function in the parenthesis is holomorphic as x, y vary on the real axis and admits the period Ω in x, y.

Let $N(x, y)$ be a function similar to $M(x, y)$, and write

$$f_1(x) = \int_0^{\prime\,\Omega} M(x,y)f_0(y)\,dy, \qquad f_2(x) = \int_0^{\prime\,\Omega} N(x,y)f_1(y)\,dy;$$

then, applying equation A19 and A20, we obtain

$$\begin{aligned} \int_0^{\prime\,\Omega} N(x,y)\,dy\int_0^{\prime\,\Omega} M(y,z)f(z)\,dz = &-\pi^2 N_1(x)M_1(x)f(x)\\ &+\int_0^{\Omega} f(z)\,dz\int_0^{\prime\,\Omega} N(x,y)M(y,z)\,dy, \end{aligned} \tag{A21}$$

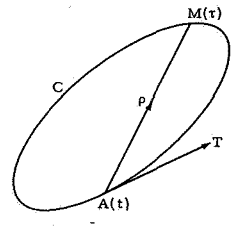
where $N_1(x)$ is the residue of $N(x, y)$ at the pole $y=x$.

Let $A(x, y)$ be an arbitrary point on C such that $x=f(t)$, $y=\varphi(t)$, and A describe a whole circuit of C in the direct sense as t varies from 0 to Ω. If $M[x(\tau), y(\tau)]$ is an attracting point and $\rho(\tau)$ is the density at M, then the tangential component along AT of attraction at A is

$$(T) = \int_0^{'\Omega} \frac{\rho(\tau)\cos(T\hat{A}M)ds}{\overline{AM}}$$

$$= \int_0^{'\Omega} \frac{x'(t)[x(\tau)-x(t)]+y'(t)[y(\tau)-y(t)]}{[x(\tau)-x(t)]^2+[y(\tau)-y(t)]^2}$$

$$\times \frac{\sqrt{x'^2(\tau)+y'^2(\tau)}}{\sqrt{x'^2(t)+y'^2(t)}}\rho(\tau)d\tau.$$



The denominator of the kernel of this integral, which we denote by $N(\tau, t)$, vanishes for $t=\tau$. Let $\tau = t+h$ and expand $x(\tau)$, $x'(\tau)$, $y(\tau)$, $y'(\tau)$ in Taylor series in powers of h at $\tau = t$; then,

$$N(\tau, t) = \frac{1}{\tau - t} + \frac{1}{2}\frac{x'x''+y'y''}{x'^2+y'^2} + \cdots.$$

Thus the singularity of the tangential derivative of the kernel for the logarithmic potential of a simple layer is a simple pole with residue 1.

Suppose that H is a continuous kernel but that K is a singular kernel with a simple pole at $y=x$, and consider the integral equation

$$f(x) = \varphi(x) + \int_0^{\Omega} H(x, y)f(y)dy + \int_0^{'\Omega} K(x, y)f(y)dy.$$

If the integral exists, then it is equal to its principal value, so that

$$f(x) = \varphi(x) + \int_0^{'\Omega} N(x, y)f(y)dy, \qquad N(x, y) = H(x, y) + K(x, y). \tag{A22}$$

Any solution of equation A22 is a solution of

$$f(x) = \varphi(x) + \int_0^{'\Omega} N(x, y)\varphi(y)dy + \int_0^{'\Omega} N(x, y)dy \int_0^{'\Omega} N(y, z)f(z)dz. \tag{A23}$$

It can be shown conversely that any solution $f(x)$ of equation A23 satisfies equation A22, owing to the theorem we have just presented. Hence the process of iteration for the integral equation containing such singular integrals is justified.

Now we return to the solution of integral equation A17. All functions appearing in the equation are holomorphic since curve C is regular analytic and periodic in s and s', except that $\sin\psi/r$ admits a simple pole of residue 1 at $s=s'$.

Let

$$\eta(s) = -\frac{\chi(s)}{\pi\alpha}, \qquad A(s, s') = \frac{1}{\pi}\frac{\cos\psi}{r}, \qquad B(s, s') = \frac{2i\omega\cos\theta}{\pi\alpha}\frac{\sin\psi}{r},$$

where $B(s, s')$ has residue $2i\omega\cos\theta/\pi\alpha$ at $s=s'$; then equation A17 is written

$$\rho(s) = \int_C A(s, s')\rho(s')ds' + \int_C' B(s, s')\rho(s')ds' + \eta(s).$$

Iterating the kernel by using equation A21 and remembering that

$$\lim_{s\to s'} (s'-s)B(s,s') = \frac{2i\omega \cos\theta}{\pi\alpha},$$

we arrive at the equation

$$\left(1-\frac{4\omega^2\cos^2\theta}{\alpha^2}\right)\rho(s) = \int_C K(s,s'')\rho(s'')ds'' + \Theta(s), \tag{A24}$$

where

$$K(s,s'') = \int_C A(s,s')A(s',s'')ds' + \int_C' A(s,s')B(s',s'')ds'$$
$$+ \int_C' B(s,s')A(s',s'')ds' + \int_C' B(s,s')B(s',s'')ds',$$
$$\Theta(s) = \eta(s) + \int_C A(s,s')ds' + \int_C' B(s,s')\eta(s')ds'.$$

The factor $1-4\omega^2\cos^2\theta/\alpha^2$ does not vanish if the ocean does not cross the critical latitude (Bertrand, 1923). Integral equation A24 is an ordinary Fredholm equation and admits a solution in general. Thus the existence of the Green function $G(x, y; \xi, \eta)$ is established.

The next problem is to integrate integral equation A15. Thus the problem of dynamical tide has been reduced to that of solving three Fredholm's integral equations in succession.

RITZ'S VARIATIONAL METHOD

Equations A6a for tide on a geographical map is seen to be

$$\left.\begin{aligned} &u+\frac{2i\omega\cos\theta}{\alpha}v = -\frac{\partial\Phi}{\partial x}, \qquad v-\frac{2i\omega\cos\theta}{\alpha}u = -\frac{\partial\Phi}{\partial y},\\ &\frac{\zeta}{k^2} = \frac{\partial(hu)}{\partial x}+\frac{\partial(hv)}{\partial y}, \qquad \alpha^2\Phi = g\zeta + \Pi'' + F(x,y),\\ &\Pi'' = -\iint \frac{\zeta' dx' dy'}{k'^2 r}, \end{aligned}\right\} \tag{A25}$$

with the boundary condition

$$hvdx - hudy = 0$$

along the boundary curve.

Poincaré (1910b) eliminated h from these equations and obtained

$$\frac{\zeta}{k^2} = \frac{\partial}{\partial x}\left(h_1\frac{\partial\Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(h_1\frac{\partial\Phi}{\partial y}\right) + i\left(\frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial\eta}{\partial x}\right),$$
$$\alpha^2\Phi = g\zeta + \Pi'' + F(x,y),$$
$$h_1 = \frac{h\alpha^2}{4\omega^2\cos^2\theta-\alpha^2}, \qquad \eta = \frac{2\omega h_1\cos\theta}{\alpha} = \frac{2\omega\alpha h\cos\theta}{4\omega^2\cos^2\theta-\alpha^2},$$

with the boundary condition

$$h\left(\frac{\partial\Phi}{\partial n} + \frac{2i\omega\cos\theta}{\alpha}\frac{\partial\Phi}{\partial s}\right) = 0.$$

Putting $\Phi=\Phi_1+i\Phi_2$, $\zeta=\zeta_1+i\zeta_2$, $\Pi''=\Pi_1''+i\Pi_2''$, $F=F_1+iF_2$ and separating the real and imaginary parts, we obtain

$$\frac{\partial}{\partial x}\left(h_1\frac{\partial\Phi_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(h_1\frac{\partial\Phi_1}{\partial y}\right)-\frac{\partial\Phi_2}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\Phi_2}{\partial y}\frac{\partial\eta}{\partial x}-\frac{\zeta_1}{k^2}=0,$$

$$\frac{\partial}{\partial x}\left(h_1\frac{\partial\Phi_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(h_1\frac{\partial\Phi_2}{\partial y}\right)+\frac{\partial\Phi_1}{\partial x}\frac{\partial\eta}{\partial y}-\frac{\partial\Phi_1}{\partial y}\frac{\partial\eta}{\partial x}-\frac{\zeta_2}{k^2}=0,$$

$$-\frac{\Phi_1}{k^2}+\frac{g\zeta_1}{\alpha^2k^2}+\frac{\Pi_1''}{\alpha^2k^2}+\frac{F_1}{\alpha^2k^2}=0,$$

$$-\frac{\Phi_2}{k^2}+\frac{g\zeta_2}{\alpha^2k^2}+\frac{\Pi_2''}{\alpha^2k^2}+\frac{F_2}{\alpha^2k^2}=0\ ,$$

with the boundary condition

$$h\left(\frac{\partial\Phi_1}{\partial n}-\frac{2\omega\cos\theta}{\alpha}\frac{\partial\Phi_2}{\partial s}\right)=0\qquad h\left(\frac{\partial\Phi_2}{\partial n}+\frac{2\omega\cos\theta}{\alpha}\frac{\partial\Phi_1}{\partial s}\right)=0.$$

If we multiply the four equations respectively by $\delta\Phi_1dxdy$, $\delta\Phi_2dxdy$, $\delta\zeta_1dxdy$, $\delta\zeta_2dxdy$ and integrate over Σ, then the integral should be zero. This integral is shown after algebraic computation to equal the variation δJ of this integral J:

$$\delta J=0,$$

$$J=\iint_\Sigma\left\{-\frac{h_1}{2}\left[\left(\frac{\partial\Phi_1}{\partial x}\right)^2+\left(\frac{\partial\Phi_1}{\partial y}\right)^2+\left(\frac{\partial\Phi_2}{\partial x}\right)^2+\left(\frac{\partial\Phi_2}{\partial y}\right)^2\right]+\eta\left(\frac{\partial\Phi_2}{\partial x}\frac{\partial\Phi_1}{\partial y}-\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial y}\right)\right.$$
$$\left.-\frac{\zeta_1\Phi_1+\zeta_2\Phi_2}{k^2}+\frac{g(\zeta_1^2+\zeta_2^2)}{2\alpha^2k^2}+\frac{\zeta_1\Pi_1''+\zeta_2\Pi_2''}{2\alpha^2k^2}+\frac{\zeta_1F_1+\zeta_2F_2}{\alpha^2k^2}\right\}dxdy. \tag{A26}$$

Blondel (1912) and Jager (1916) discussed special cases of this integral of Poincaré (1910b). This integral contains h and η, and it becomes infinite at the critical latitude. In order to avoid this difficulty, Bertrand (1923) transformed equation A25 by putting

$$u=u_1+iu_2,\qquad v=v_1+iv_2,$$

and

$$u_1-\frac{2\omega\cos\theta}{\alpha}v_2+\frac{\partial\Phi_1}{\partial x}=0,\qquad u_2+\frac{2\omega\cos\theta}{\alpha}v_1+\frac{\partial\Phi_2}{\partial x}=0,$$

$$v_1+\frac{2\omega\cos\theta}{\alpha}u_2+\frac{\partial\Phi_1}{\partial y}=0,\qquad v_2-\frac{2\omega\cos\theta}{\alpha}u_1+\frac{\partial\Phi_2}{\partial y}=0,$$

$$\frac{\partial(hu_1)}{\partial x}+\frac{\partial(hv_1)}{\partial y}-\frac{\zeta_1}{k^2}=0,\qquad \frac{\partial(hu_2)}{\partial x}+\frac{\partial(hv_2)}{\partial y}-\frac{\zeta_2}{k^2}=0,$$

$$-\frac{\Phi_1}{k^2}+\frac{g\zeta_1}{\alpha^2k^2}+\frac{\Pi_1''}{\alpha^2k^2}+\frac{F_1}{\alpha^2k^2}=0,\qquad -\frac{\Phi_2}{k^2}+\frac{g\zeta_2}{\alpha^2k^2}+\frac{\Pi_2''}{\alpha^2k^2}+\frac{F_2}{\alpha^2k^2}=0,$$

with the boundary condition

$$hu_1dy - hv_1dx = 0, \qquad hu_2dy - hv_2dx = 0.$$

If we multiply these eight equations respectively by $-h\delta u_1$, $-h\delta u_2$, $-h\delta v_1$, $-h\delta v_2$, $\delta\Phi_1$, $\delta\Phi_2$, $\delta\zeta_1$, $\delta\zeta_2$ and integrate over the whole surface Σ, then we obtain

$$\delta J_1 = 0,$$

$$\begin{aligned} J_1 = &-\iint_\Sigma \frac{h}{2}\left[u_1^2+v_1^2+u_2^2+v_2^2-\frac{4\omega\cos\theta}{2}(u_1v_2-u_2v_1)\right]dxdy \\ &+\iint_\Sigma\left[\Phi_1\left(\frac{\partial(hu_1)}{\partial x}+\frac{\partial(hv_1)}{\partial y}\right)+\Phi_2\left(\frac{\partial(hu_2)}{\partial x}+\frac{\partial(hv_2)}{\partial y}\right)\right]dxdy \\ &+\iint_\Sigma\left[-\frac{\zeta_1\Phi_1+\zeta_2\Phi_2}{k^2}+\frac{g(\zeta_1^2+\zeta_2^2)}{2\alpha^2k^2}+\frac{\Pi_1''\zeta_1+\Pi_2''\zeta_2}{2\alpha^2k^2}+\frac{F_1\zeta_1+F_2\zeta_2}{\alpha^2k^2}\right]dxdy. \end{aligned}$$

The integral J_1 does not vanish.

Another form of the integral can be obtained if we eliminate Φ and ζ from J_1 (Bertrand, 1923).

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